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STUDY OF PERTURBED PERIODIC SYSTEMS OF DIFFERENTIAL EQUATIONS - THE STROBOSCOPIC METHOD

by C. M. Petty, W. E. Johnson, R. J. Dickson, A. Feinstein, and T. R. Jenkins

Prepared by
LOCKHEED AIRCRAFT CORPORATION
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CONTENTS

1. INTRODUCTION

The exact statement of hypotheses and results are given in the following sections. Here, we will discuss the problems in general terms. The class of differential equations which will occupy our attention is of the form

$$u' = F(u, \varepsilon, t) \tag{1.1}$$

where u is an N-vector, ε a small parameter and t the independent variable. The function $F(u, \varepsilon, t)$ is assumed to be periodic in t of period $T > 0$ for all u and ε and the unperturbed equation

$$z' = F(z, 0, t) \tag{1.2}$$

is assumed to have the property that all its solutions are periodic of period T.

If $u(x, \varepsilon, t)$ is the solution of (1.1) with initial condition $u(x, \varepsilon, 0) = x$, then we define the transformation $h(x, \varepsilon)$ with the property $h(x, 0) = x$ by

$$h(x, \varepsilon) = u(x, \varepsilon, T). \tag{1.3}$$

Since $F(u, \varepsilon, t)$ is periodic in t of period T, it follows that the iterates of the transformation $h(x, \varepsilon)$ yield a sequence of points which are the values of $u(x, \varepsilon, t)$ at positive integral multiples of T. A fixed point of $h(x, \varepsilon)$ corresponds to a periodic solution of (1.1) of period T and a fixed point of the iterate $h^{(n)}(x, \varepsilon)$ corresponds to a solution of period nT (not necessarily the minimal period). In fact, the transformation $h(x, \varepsilon)$ determines virtually all the important qualitative characteristics of the solutions to (1.1).

An autonomous system of the form

$$v' = g(v, \varepsilon) , \quad g(v, 0) = 0 \tag{1.4}$$

with the property that the solution $v(x, \varepsilon, t)$, with initial condition $v(x, \varepsilon, 0) = x$, satisfies $v(x, \varepsilon, T) = h(x, \varepsilon)$ will be called a stroboscopic differential equation of the system (1.1). Thus, the solutions of (1.1) and (1.4) with the same initial conditions at $t = 0$ coincide when t is an integral multiple of T. The principal problem of our investigation is the determination of a stroboscopic differential equation of (1.1) and a transformation $\varphi(y, \varepsilon, t)$ with the properties

$$\begin{cases} \varphi(y, \varepsilon, 0) = y , \\ \varphi(y, \varepsilon, t+T) = \varphi(y, \varepsilon, t) , \end{cases} \tag{1.5}$$

such that if $v(x, \varepsilon, t)$ is a solution to (1.4) then

$$u = \varphi(v(x, \varepsilon, t), \varepsilon, t)$$

is a solution to (1.1).

The "stroboscopic method" was invented by N. Minorsky, see [1, pp. 390-415], in collaboration with M. Schiffer. Briefly, it takes the following form. Let (1.1) be of the type

$$u' = \varepsilon f(u, \varepsilon, t) . \tag{1.6}$$

Then, in place of (1.4), the averaged equation

$$\begin{cases} v' = \varepsilon F(v) \\ F(v) = \frac{1}{T} \int_0^T f(u, 0, t)dt \end{cases} \tag{1.7}$$

is used. Questions of the existence and stability of periodic solutions with period T to (1.6) can then be formulated in terms of the behavior of the solutions to (1.7). In our treatment here, we seek convergence of a series in which the first term is the above averaged equation. Yorinaga [2] and Urabe [3] have elaborated on Minorsky's method and have thereby placed it on a more rigorous footing.

Sharshanov [4], [5] and Urabe [3], [6], [7] have considered these problems without dependence on ε. In [4], [5], [6] and [7], they start with the transformation $h(x)$ given. In [3] the starting point is a periodic non-autonomous system. They give sufficient conditions for the existence of an analytic autonomous system. By extending our system to a system of one higher dimension we can transform our problem into their setting. But their hypotheses exclude the function F and h which satisfy our conditions. The principal difference is that for small ε, $h(x, \varepsilon)$ is a small perturbation of the identity transformation but this case is specifically excluded in their hypotheses.

In section 2, we give the formal computation of the stroboscopic differential equation and the transformation $\varphi(y, \varepsilon, t)$. In section 3, certain special systems are investigated for which we not only give exact convergence theorems for the stroboscopic representation of the solutions but also more specific information regarding the form and qualitative behavior of the solutions. In section 4, the formal part of the general theory is developed and in section 5 a convergence theorem is stated and proved.

During the early stages of the investigation, some casual discussion of perturbation theory with a physicist colleague resulted in the resolution of a mathematical question arising in Quantum Chemistry. Although the subject matter of the subsequent publication is peripheral to the contract study, an acknowledgement was included and a reprint of the paper is reproduced as an Appendix to this report.

2. STROBOSCOPIC REPRESENTATION

We rewrite (1.1) with N-vector y in place of u:

$$y' = F(y, \varepsilon, t) . \tag{2.1}$$

Our interest is in real solutions to (2.1) and consequently we always assume that F is real when y, ε and t are real. However, we introduce assumptions over complex domains in y and ε in order to utilize analytic expansion theorems. Let E^N be N-dimensional (real) Euclidean space and let C^N be N-dimensional complex vector space. Thus C^N is homeomorphic to E^{2N}. Let H be an open region of C^{N+1} which includes all points $(x_1, \ldots, x_N, 0)$ where the x_i are real. We assume that $F(y, \varepsilon, t)$ is continuous in the $(N+2)$ variables (y, ε, t) for all $(y, \varepsilon) \in H$ and $t \in E^1$. Also, for each fixed t, $t \in E^1$, we assume that F is analytic in (y, ε) over H.

These conditions imply the local existence and uniqueness of the solution $y(x, \varepsilon, t)$ with initial condition $y(x, \varepsilon, 0) = x$, $(x, \varepsilon) \in H$. For some fixed $T > 0$ we assume

$$F(y, \varepsilon, t+T) = F(y, \varepsilon, t), \ (y, \varepsilon) \in H, \ t \in E^1. \tag{2.2}$$

Finally, we assume that the unperturbed equation

$$z' = F(z, 0, t) \tag{2.3}$$

has the property that all solutions $z(x, t)$, with $z(x, 0) = x$, $x \in H_0 = \{x \in C^N | (x, 0) \in H\}$ are periodic of period T, i.e.

$$z(x, t+T) = z(x, t), \ x \in H_0, \ t \in E^1 \tag{2.4}$$

In particular, $z(x, t) \in H_0$ for $x \in H_0$, $t \in E^1$.

A problem of interest by itself is to find additional assumptions which may be imposed on (2.3) directly in order to insure that all solutions are periodic of period T. If we assume that each solution $z(x, t)$ may be continued to all t, then one such condition is the existence of a real constant c such that

$$F(z, 0, c+t) = -F(z, 0, c-t)$$

for all $z \in H_0$, $t \in E^1$. This is a generalization of a known result for linear equations, see Epstein [8, p. 691]. To prove this assertion, let $z(t)$ be any solution to (2.3), then $z(2c-t)$ is also a solution and it has the same value as $z(t)$ at $t = c$. Consequently, by uniqueness, $z(t) = z(2c-t)$ or $z(c-t) = z(c+t)$. On the other hand, because of (2.2), $q(t) = z(t+T)$ is also a solution and

$$q(c - \frac{T}{2}) = z(c + \frac{T}{2}) = z(c - \frac{T}{2})$$

and again, by uniqueness, $q(t) = z(t) = z(t+T)$. Another condition for all solutions to be periodic will be given in Section 3.

The condition (2.4) enables us to perform a reduction of (2.1). From the analytic expansion theorem, see [9, p. 36], $z(x, t)$ is continuous in (x, t) over $H_0 \times E^1$ and for each t, $t \in E^1$, $z(x, t)$ is analytic in x over H_0. Each element of the Jacobian matrix

$$J(x, t) = \frac{\partial z(x,t)}{\partial x}$$

has these same properties; the continuity following from Cauchy's integral formula for a derivative. By (2.3), $J(x, t)$ satisfies a linear matrix differential equation in the independent variable t and $J(x, 0)$, $x \in H_0$, is the identity matrix. Consequently, $J(x, t)$ is non-singular for all $x \in H_0$, $t \in E^1$ (see Bellman [10, p. 10]). There exists a countable collection of open sets G_n contained in C^N such that $H_0 = \bigcup_n G_n$ and such that the closure $\bar{G}_n$ is compact and $\bar{G}_n \subset H_0$. Due to (2.4), the set $A_n = z(\bar{G}_n, E^1)$

is compact and $A_n \subset H_0$. Consequently, there exist $\delta_n > 0$ such that if $S_n = \{\varepsilon \mid |\varepsilon| < \delta_n\}$ then $A_n \times S_n \subset H$. Now let

$$G = \bigcup_n (G_n \times S_n) \tag{2.5}$$

where G is then an open subset of H and $H_0 \times \{0\} \subset G$. Consider the function

$$f(u, \varepsilon, t) = J^{-1}(u, t)\,[F(z(u, t), \varepsilon, t) - F(z(u, t), 0, t)]$$

which has the properties that $f(u, \varepsilon, t)$ is continuous in (u, ε, t) for all $(u, \varepsilon) \in G$ and $t \in E^1$ and for each t, f is analytic in (u, ε) over G.

The differential equation

$$u' = f(u, \varepsilon, t) \tag{2.6}$$

has all the properties imposed on (2.1), where H is replaced by the subregion G, and in addition

$$f(u, 0, t) = 0\ ,\ u \in H_0,\ t \in E' \ . \tag{2.7}$$

If $u(x, \varepsilon, t)$ is the solution of (2.6) with the initial condition $u(x, \varepsilon, 0) = x$, $(x, \varepsilon) \in G$, then $y = z(u(x, \varepsilon, t), t)$ is the solution to (2.1) with the same initial condition.

Let S be an arbitrary compact set in H_0. Then, due to (2.7) and the periodicity condition of $f(u, \varepsilon, t)$ in t, we conclude that for $|\varepsilon|$ sufficiently small the solutions $u(x, \varepsilon, t)$ with $x \in S$ may be extended to an interval in t which includes $[0, T]$. Consequently, by the analytic expansion theorem, the transformation function $h(x, \varepsilon) = u(x, \varepsilon, T)$ is defined and analytic over an open subset V of G and such that $H_0 \times \{0\} \subset V$. The transformation $h(x, \varepsilon)$ has the property that $h(x, 0) = x$ for $x \in H_0$. It may be observed that the transformation function $y(x, \varepsilon, T)$ of (2.1) is identical to $h(x, \varepsilon)$ over V since $y(x, \varepsilon, T) = z(u(x, \varepsilon, T), T) = z(u(x, \varepsilon, T), 0) = u(x, \varepsilon, T)$.

We now define the stroboscopic representation of the solutions to (2.6). Suppose there exist N-vector valued functions $g(x, \varepsilon)$, $\varphi(x, \varepsilon, t)$, where x is an N-vector, with the following properties. There exists an open subset G_0 of G such that $H_0 \times \{0\} \subset G_0$ and $g(x, \varepsilon)$ is analytic in (x, ε) over G_0. Also

$$(2.8) \qquad g(x, 0) = 0 , \quad x \in H_0 .$$

The functions $\varphi(x, \varepsilon, t)$ and $\frac{\partial\varphi}{\partial t}$ are continuous in (x, ε, t) for $(x,\varepsilon) \in G_0$ and $t \in E^1$ and for each t, $t \in E^1$, these functions are analytic in (x, ε) over G_0. In addition

$$(2.9) \qquad \varphi(x, \varepsilon, t+T) = \varphi(x, \varepsilon, t) , \ (x, \varepsilon) \in G_0, \ t \in E^1,$$

$$(2.10) \qquad \varphi(x, \varepsilon, 0) = x \qquad , \ (x, \varepsilon) \in G_0 ,$$

$$(2.11) \qquad \varphi(x, 0, t) = x \qquad , \ (x, t) \in H_0 \times E^1 .$$

Finally, if $v(x, \varepsilon, t)$ is the solution to the autonomous equation

$$(2.12) \qquad v' = g(v, \varepsilon)$$

with initial condition $v(x, \varepsilon, 0) = x$, $(x, \varepsilon) \in G_0$, then

$$(2.13) \qquad u(x, \varepsilon, t) = \varphi(v(x, \varepsilon, t), \varepsilon, t)$$

is the solution of (2.6) with the same initial condition.

As in the discussion of the transformation function $h(x, \varepsilon)$ for (2.6), the condition (2.8) implies that $v(x, \varepsilon, T)$ is defined and analytic in (x, ε) over some open subset V_0 of G_0 and such that $H_0 \times \{0\} \subset V_0$. The above conditions then imply that $h(x,\varepsilon) = v(x, \varepsilon, T)$ over V_0 and consequently (2.12) is a stroboscopic differential equation of (2.6) and therefore also of (2.1).

We will show that if there exists a stroboscopic representation of the solutions of (2.6), as defined above, then the functions $g(x, \varepsilon)$ and $\varphi(x, \varepsilon, t)$

are uniquely determined. We do this by exhibiting an explicit calculation of these functions in terms of $f(u, \varepsilon, t)$.

Substitute (2.13) into (2.6), denoting the Jacobian matrix of $\varphi(x, \varepsilon, t)$ with respect to x by $\varphi_{,1}(x, \varepsilon, t)$, we have

$$(2.14) \quad \varphi_{,1}(v, \varepsilon, t)\, g(v, \varepsilon) + \frac{\partial\varphi}{\partial t}(v, \varepsilon, t) = f(\varphi(v, \varepsilon, t), \varepsilon, t) .$$

We now define open subsets $G_i (i = 1, 2, 3)$ of G_0 such that $G_0 \supset G_1 \supset G_2 \supset G_3$ and G_3 contains $H_0 \times \{0\}$. Introducing the dummy variable y, we will show that

$$(2.15) \quad \varphi_{,1}(y, \varepsilon, t)\, g(y, \varepsilon) + \frac{\partial\varphi}{\partial t}(y, \varepsilon, t) = f(\varphi(y, \varepsilon, t), \varepsilon, t)$$

is an identity for all $t \in E^1$ and all $(y, \varepsilon) \in G_3$. For each t, $t \in E^1$, the left-hand side of (2.15) is analytic in (y, ε) over G_0. Due to the properties of $\varphi(y, \varepsilon, t)$, there exists an open subset G_1 of G_0 such that $(\varphi(y, \varepsilon, t), \varepsilon) \in G$ for all $(y, \varepsilon) \in G_1$ and $t \in E^1$. Consequently, for each t, $t \in E^1$, both sides of (2.15) are analytic in (y, ε) over G_1. The existence of G_2 and G_3, defined below, follows from arguments similar to those applied to $h(x, \varepsilon)$ for equation (2.6). Let G_2 be an open subset of G_1, such that if $(x, \varepsilon) \in G_2$, then $v(x, \varepsilon, t)$ may be extended to all t in $[0, T]$ and $(v(x, \varepsilon, t), \varepsilon) \in G_1$ for all $t \in [0, T]$. Let G_3 be an open subset of G_2, such that if $(x, \varepsilon) \in G_3$, then $v(x, \varepsilon, t)$ may be extended to all t in $[-T, T]$ and $(v(x, \varepsilon, t), \varepsilon) \in G_2$ for all $t \in [-T, T]$. Now let $(y, \varepsilon) \in G_3$ and $t_0 \varepsilon [0, T]$. If $x = v(y, \varepsilon, -t_0)$, then $(x, \varepsilon) \in G_2$. We apply (2.14) to the solution $v(x, \varepsilon, t)$ and observe that $v(x, \varepsilon, t_0) = y$ since the equation (2.12) is autonomous. Thus (2.15) is an identity for $(y, \varepsilon) \in G_3$ and $t \in [0, T]$. But since each side of (2.15) is periodic in t of period T, this result may be extended to all $t \in E^1$.

From the properties of f, g and φ we have

$$(2.16) \qquad f(y, \varepsilon, t) = \sum_{n=1}^{\infty} \varepsilon^n f_n(y, t)$$

$$
(2.17)\quad \begin{cases} g(y,\ \varepsilon) = \sum_{n=1}^{\infty} \varepsilon^n g_n(y) \\ \varphi(y,\ \varepsilon,\ t) = y + \sum_{n=1}^{\infty} \varepsilon^n \varphi_n(y,\ t) \end{cases}
$$

where the f_n and φ_n are periodic in t of period T and $\varphi_n(y,\ 0) = 0$, $n = 1, 2, 3 \ldots$.

In order to expand the right-hand side of (2.15) as a power series in ε, we need certain preliminary expansion formulas for analytic functions. Some of these will be used in later sections. Let x, η be N-vectors with components $x = (x^{(1)}, \ldots, x^{(N)})$, $\eta = (\eta^{(1)}, \ldots, \eta^{(N)})$ and let $f(x)$ be a N-vector valued function of x. A known result is

$$
(2.18)\qquad f(x+\eta) = \sum_{k=0}^{\infty} \frac{1}{k!} f_k(x,\ \eta)
$$

where

$$
(2.19)\quad \begin{cases} f_k(x,\ \eta) = \sum_{i_1,\ldots,i_k=1}^{N} \eta^{(i_1)} \cdots \eta^{(i_k)} \dfrac{\partial^k f(x)}{\partial x^{(i_1)} \cdots \partial x^{(i_k)}} \qquad k = 1,2,\ldots \\ f_0(x,\ \eta) = f(x) \end{cases}
$$

Now let $\eta = \sum_{j=1}^{\infty} \varepsilon^j \eta_j$, $\eta_j = (\eta_j^{(1)}, \ldots, \eta_j^{(N)})$.

Then

$$f_k(x, \eta) = \sum_{i_1,\ldots,i_k=1}^{N} \left(\sum_{j=1}^{\infty} \varepsilon^j \eta_j^{(i_1)} \right) \cdots \left(\sum_{j=1}^{\infty} \varepsilon^j \eta_j^{(i_k)} \right. \frac{\partial^k f(x)}{\partial x^{(i_1)} \cdots \partial x^{(i_k)}}$$

$$= \sum_{i_1,\ldots,i_k=1}^{N} \sum_{j_1,\ldots,j_k=1}^{\infty} \varepsilon^{j_1+\cdots+j_k} \eta_{j_1}^{(i_1)} \cdots \eta_{j_k}^{(i_k)} \frac{\partial^k f(x)}{\partial x^{(i_1)} \cdots \partial x^{(i_k)}}$$

$$= \sum_{j_1,\ldots,j_k=1}^{\infty} \varepsilon^{j_1+\cdots+j_k} f_k(x, \eta_{j_1}, \ldots, \eta_{j_k})$$

where

$$f_k(x, \eta_{j_1}, \ldots, \eta_{j_k}) = \sum_{i_1,\ldots,i_k=1}^{N} \eta_{j_1}^{(i_1)} \cdots \eta_{j_k}^{(i_k)} \frac{\partial^k f(x)}{\partial x^{(i_1)} \cdots \partial x^{(i_k)}}$$

This yields

$$(2.20) \quad f_k(x, \sum_{j=1}^{\infty} \varepsilon^j \eta_j) = \sum_{\nu=k}^{\infty} \varepsilon^\nu \sum_{j_1+\cdots+j_k=\nu} f_k(x, \eta_{j_1}, \ldots, \eta_{j_k})$$

for $k = 1, 2, 3 \ldots$. Throughout this paper a summation symbol of the type $\sum_{j_1+\cdots+j_k=\nu}$ will be used to denote a summation over all k-vectors $(j_1,\ldots,j_k)$ whose elements are positive integers such that $j_1 + \cdots + j_k = \nu$.

From (2.18) and (2.20) we have

$$(2.21)\quad f(x+\sum_{j=1}^{\infty}\varepsilon^j\eta_j) = f(x) + \sum_{\nu=1}^{\infty}\varepsilon^{\nu}\sum_{k=1}^{\nu}\frac{1}{k!}\sum_{j_1+\dots+j_k=\nu} f_k(x,\ \eta_{j_1},\dots,\eta_{j_k})$$

Consider the N-vector valued functions of the form

$$(2.22)\qquad \begin{cases} h(x,\ \varepsilon) = x + \sum_{n=1}^{\infty} \varepsilon^n h_n(x) \\ \ell(x,\ \varepsilon) = x + \sum_{n=1}^{\infty} \varepsilon^n \ell_n(x) \end{cases}$$

where $h_n(x)$ and $\ell_n(x)$ are analytic and $h_n = (h_n^{(1)},\ \dots,\ h_n^{(N)})$, $\ell_n = (\ell_n^{(1)},\ \dots,\ \dots,\ \ell_n^{(N)})$. Then

$$h(\ell(x,\ \varepsilon),\ \varepsilon) = x + \sum_{n=1}^{\infty}\varepsilon^n\ell_n(x) + \sum_{n=1}^{\infty}\varepsilon^n h_n\left(x + \sum_{j=1}^{\infty}\varepsilon^j\ell_j(x)\right)$$

From (2.21), we have

$$h_n(x+\sum_{j=1}^{\infty}\varepsilon^j\ell_j(x)) = h_n(x)$$

$$+\sum_{\nu=1}^{\infty}\varepsilon^{\nu}\sum_{k=1}^{\nu}\frac{1}{k!}\sum_{j_1+\dots+j_k=\nu} h_{nk}(x,\ \ell_{j_1}(x),\ \dots,\ell_{j_k}(x))$$

where

$$h_{nk}(x,\ \ell_{j_1}(x),\ \dots,\ \ell_{j_k}(x)) = \sum_{i_1,\dots,i_k=1}^{N} \ell_{j_1}^{(i_1)}(x)\dots\ell_{j_k}^{(i_k)}(x)\,\frac{\partial^k h_n(x)}{\partial x^{(i_1)}\dots\partial x^{(i_k)}}$$

Let

$$L_{n\nu}(x) = \sum_{k=1}^{\nu} \frac{1}{k!} \sum_{j_1+\dots+j_k=\nu} h_{nk}(x, \ell_{j_1}(x), \dots, \ell_{j_k}(x))$$

then, after some manipulation, we obtain

$$(2.23)\quad h(\ell(x, \varepsilon), \varepsilon) = x + \sum_{n=1}^{\infty} \varepsilon^n [\ell_n(x) + h_n(x)] + \sum_{n=2}^{\infty} \varepsilon^n \sum_{\nu=1}^{n-1} L_{n-\nu,\nu}(x)$$

Returning to (2.15), it is seen from (2.16) and (2.17) that for fixed t the functions $\ell(y, \varepsilon) = \varphi(y, \varepsilon, t)$ and $h(y, \varepsilon) = y + f(y, \varepsilon, t)$ have the same form as (2.22). Hence $h(\varphi(y, \varepsilon, t), \varepsilon)$ can be obtained from (2.23) and then $f(\varphi(y, \varepsilon, t), \varepsilon, t) = h(\varphi(y, \varepsilon, t), \varepsilon) - \varphi(y, \varepsilon, t)$. This yields

$$(2.24)\quad f(\varphi(y, \varepsilon, t), \varepsilon, t) = \sum_{n=1}^{\infty} \varepsilon^n f_n(y, t) + \sum_{n=2}^{\infty} \varepsilon^n \sum_{\nu=1}^{n-1} \Phi_{n-\nu,\nu}(y, t)$$

where

$$(2.25)\quad \Phi_{m\nu}(y,t) = \sum_{k=1}^{\nu} \frac{1}{k!} \sum_{j_1+\dots+j_k=\nu} \sum_{i_1,\dots,i_k=1}^{N} \varphi_{j_1}^{(i_1)}(y,t)\dots\varphi_{j_k}^{(i_k)}(y,t) \frac{\partial^k f_m(y,t)}{\partial y^{(i_1)}\dots\partial y^{(i_k)}}$$

We now expand the left-hand side of (2.15) and for the first term we have

$$(2.26)\quad \varphi_{,1}(y,\ \varepsilon,\ t)\ g(y,\ \varepsilon) = \left[I + \sum_{n=1}^{\infty} \varepsilon^n \varphi_{n,1}(y,\ t)\right] \sum_{m=1}^{\infty} \varepsilon^m g_m(y)$$

$$= \sum_{n=1}^{\infty} \varepsilon^n g_n(y) + \sum_{n=2}^{\infty} \varepsilon^n \sum_{\nu=1}^{n-1} \varphi_{\nu,1}(y,\ t) g_{n-\nu}(y)$$

Substituting (2.24) and (2.26) into (2.15) we get

$$\frac{\partial \varphi_1}{\partial t}(y,t) = f_1(y,\ t) - g_1(y),$$

$$\frac{\partial \varphi_n}{\partial t}(y,t) = f_n(y,\ t) + \sum_{\nu=1}^{n-1} \Phi_{n-\nu,\nu}(y,t)$$

$$-g_n(y) - \sum_{\nu=1}^{n-1} \varphi_{\nu,1}(y,\ t)\ g_{n-\nu}(y)$$

$$n = 2,\ 3,\ 4,\ \ldots$$

The condition that $\varphi_n(y,\ t)$ is periodic in t of period T uniquely determines $g_n(y)$ and we have

$$(2.27)\qquad \begin{cases} g_1(y) = \dfrac{1}{T} \displaystyle\int_0^T f_1(y,\ t)dt \\[2ex] g_n(y) = \dfrac{1}{T} \displaystyle\int_0^T B_n(y,\ t)dt,\ n = 2,\ 3,\ \ldots \end{cases}$$

where

$$B_n(y, t) = f_n(y, t) + \sum_{\nu=1}^{n-1} \Phi_{n-\nu,\nu}(y, t)$$

$$- \sum_{\nu=1}^{n-1} \varphi_{\nu,1}(y, t)\, g_{n-\nu}(y) .$$

Finally, the initial condition $\varphi_n(y, 0) = 0$ uniquely determines $\varphi_n(y, t)$ and we have

$$(2.28) \qquad \begin{cases} \varphi_1(y, t) = \int_0^t [f_1(y, \tau) - g(y)] \; d\tau \\ \varphi_n(y, t) = \int_0^t [B_n(y, \tau) - g_n(y)] \, d\tau, \; n = 2, 3, \dots \end{cases}$$

It may be observed that if we define g_n and φ_n by (2.27) and (2.28), then from the assumptions on $f(y, \varepsilon, t)$ the function $g_n(y)$ is analytic in y over H_0 and, for each t , $\varphi_n(y, t)$ is analytic in y over H_0. Also φ_n and its derivative with respect to t are continuous in (y, t) over $H_0 \times E^1$.

In an applied problem yielding (2.1), the solutions $z(x, t)$ of (2.3) are quite likely known so that (2.6) is explicitly obtained. One then computes $\varphi_1, \dots, \varphi_n$ and $g_1, \dots, g_{n+1}$ from (2.27) and (2.28), i.e. the g's are computed to one higher order than the φ's. This is due to the fact that accuracy is more important in (2.12), which in general has non-periodic solutions, than in the expansion of φ in (2.17) where the φ_n are periodic in t. One then solves (2.12) on a computing machine, where a large step size is feasible due to (2.8) and the fact that (2.12) is autonomous. For long time intervals one would expect greater accuracy by this method than one would obtain by solving (2.1) or even (2.6) on the machine since "short" period oscillaations occur due to the presence of t.

3. SPECIAL EQUATIONS

In this section we will consider three special classes of equations of the form (2.6) and their stroboscopic representations.

Let $A(t)$ be a real $N \times N$ matrix continuous and periodic in t of period $T > 0$. Consider the vector-matrix equation

$$u' = \varepsilon A(t) u \quad , \quad (u, \varepsilon) \in G = C^{N+1} \tag{3.1}$$

From (2.27) and (2.28), it follows by induction that $g_n(y)$ and $\varphi_n(y, t)$ have the form $g_n(y) = D_n y$ and $\varphi_n(y, t) = Q_n(t)y$ where D_n is a constant matrix and $Q_n(t)$ is a periodic matrix in t of period T and has a continuous derivative. The D_n and $Q_n(t)$ are given by

$$\begin{cases} D_1 = \frac{1}{T} \int_0^T A(t)\, dt \\ D_n = \frac{1}{T} \int_0^T \left[A(t)\, Q_{n-1}(t) - \sum_{\nu=1}^{n-1} Q_\nu(t)\, D_{n-\nu} \right] dt \end{cases} \tag{3.2}$$

$$n = 2, 3, \ldots$$

and

$$\begin{cases} Q_1(t) = \int_0^t [A(\tau) - D_1]\, d\tau \\ Q_n(t) = \int_0^t [A(\tau) Q_{n-1}(\tau) - \sum_{\nu=1}^{n-1} Q_\nu(\tau)\, D_{n-\nu} - D_n]\, d\tau \end{cases} \tag{3.3}$$

$$n = 2, 3, \ldots$$

It is not obvious from the calculations (3.2) and (3.3) that the series

$$(3.4) \qquad \begin{cases} g(y, \varepsilon) = \sum_{n=1}^{\infty} \varepsilon^n D_n y \\ \\ \varphi(y, \varepsilon, t) = y + \sum_{n=1}^{\infty} \varepsilon^n Q_n(t) y \end{cases}$$

converge for $\varepsilon \neq 0$. However, we will now show that this is the case.

From the theory of linear equations (see [10, Chapter 1]), the solutions $u(x, \varepsilon, t)$of (3.1) exist for all t and have the form

$$(3.5) \qquad u(x, \varepsilon, t) = Y(\varepsilon, t)x \ , \ Y(\varepsilon, 0) = I$$

where $Y(\varepsilon, t)$ is an $N \times N$ matrix. From the analytic expansion theorem, for each t, $t \in E^1$, $Y(\varepsilon, t)$ is analytic in ε over C^1. In particular, since $Y(0, t) = I$, we have

$$(3.6) \qquad Y(\varepsilon, T) = I + \sum_{n=1}^{\infty} \varepsilon^n A_n \ , \quad \varepsilon \in C^1$$

where A_n $(n = 1, 2, 3 \ldots)$ is a constant matrix. For the norm of an $N \times N$ matrix $A = [a_{rs}]$ with complex elements, we choose $\|A\| = N \max |a_{ij}|$. Then $\|AB\| \leq \|A\| \ \|B\|$.

The logarithm of a matrix and the exponential matrix are defined by power series expansions. In particular, for a matrix of the form $I + B$ we use

$$\log (I + B) = B - \frac{B^2}{2} + \frac{B^3}{3} - \ldots$$

There exists $\delta > 0$ such that

$$\| \sum_{n=1}^{\infty} \varepsilon^n A_n \| < 1$$

for all $\varepsilon \in C^1$ with $|\varepsilon| < \delta$. The matrix $D(\varepsilon)$ given by

$$D(\varepsilon) = \frac{1}{T} \log \left(I + \sum_{n=1}^{\infty} \varepsilon^n A_n\right) \tag{3.7}$$

is uniquely defined and analytic in ε over $S_\delta = \{\varepsilon \in C^1 \mid |\varepsilon| < \delta\}$. Also, $D(0)$ is the zero matrix and $D(\varepsilon)$ is real for real $\varepsilon \in S_\delta$. We then have, by (3.7) and (3.6),

$$Y(\varepsilon, T) = e^{D(\varepsilon)T}, \quad \varepsilon \in S_\delta \tag{3.8}$$

and also the identity

$$Y(\varepsilon, t+T) = Y(\varepsilon, t)\, Y(\varepsilon, T), \quad (\varepsilon, t) \in C^1 \times E^1 \tag{3.9}$$

which follows in the linear theory from the periodicity of $A(t)$. Now define

$$\begin{cases} g^*(y, \varepsilon) = D(\varepsilon)\, y, \\ \varphi^*(y, \varepsilon, t) = Y(\varepsilon, t) e^{-D(\varepsilon)t}\, y. \end{cases} \tag{3.10}$$

which for each t, $t \in E^1$, are analytic in (y, ε) over $C^N \times S_\delta$. The function φ^* satisfies (2.9), (2.10) and (2.11) where the property (2.9) follows from (3.8) and (3.9). Now let $v(x, \varepsilon, t) = e^{D(\varepsilon)t}x$ be the solution of $v' = D(\varepsilon)\, v$, then

$$u(x, \varepsilon, t) = \varphi^*(v(x, \varepsilon, t), \varepsilon, t)$$

is the solution (3.5) of (3.1). Thus, the functions in (3.10) satisfy all the properties for the stroboscopic representation of the solutions to (3.1) and therefore the corresponding functions in (3.10) and (3.4) must be identical for all $t \in E^1$ and all $(y, \varepsilon) \in G_0 = C^N \times S_\delta$.

We observe that even for linear equations there may be restriction on $|\varepsilon|$ for the stroboscopic representation. The Floquet theory states that the solutions of

$$u' = A(t)u , \quad A(t + T) = A(t)$$

may always be expressed in the form

$$u(x, t) = Q(t)e^{Dt}x , \quad Q(0) = I$$

where $Q(t)$ is periodic of period T and D is a constant matrix. But it is not always possible to choose D to be a real matrix (see [9, p. 81]) and consequently, in this case, there must be a restriction on $|\varepsilon|$ for the stroboscopic representations of the solutions to (3.1).

Next we consider equations of the form

$$\text{(3.11)} \qquad u' = \varepsilon f(t) F(u)$$

where $f(t)$ is a continuous, periodic scalar function of t of period $T > 0$ and $F(u)$ is an N-vector valued function analytic over an open set $H_0 \subset C^N$. Thus, for each t, the right-hand side of (3.11) is analytic in (u, ε) over $G = H_0 \times C^1$. Let $w(x, t)$ be the solution of the initial value problem

$$\text{(3.12)} \qquad w' = F(w) , \quad w(x, 0) = x.$$

It is known that there exist N-vector valued functions $H_n(x)$ $(n = 1, 2, \ldots)$ which are analytic on H_0, and for each $x_0 \in H_0$ there exists $\delta(x_0) > 0$, and $\rho(x_0) > 0$ such that

$$\text{(3.13)} \qquad w(x, t) = x + \sum_{n=1}^{\infty} t^n H_n(x)$$

converges for $\|x-x_0\| < \rho(x_0)$ and $|t| < \delta(x_0)$.

The functions $H_n(x)$, with $H_1(x) = F(x)$, may be calculated from $F(x)$ by successive application of a differential operator. Since (3.12) is autonomous, if $|t_1|$, $|t_2|$ and $|t_1 + t_2|$ are all less than $\delta(x)$ then

(3.14) $$w(w(x, t_1), t_2) = w(x, t_1 + t_2)$$

Now let

(3.15) $$\begin{cases} m = \frac{1}{T} \int_0^T f(\tau)\, d\tau \\ a(t) = \int_0^t f(\tau)\, d\tau - mt \\ M = \sup\limits_{0 \le t \le T} |a(t)| \end{cases}$$

where $a(t)$ is periodic of period T. We define

(3.16) $$\varphi(y, \varepsilon, t) = y + \sum_{n=1}^{\infty} \varepsilon^n a(t)^n H_n(y) .$$

Let $G_0 = \{(x, \varepsilon) \varepsilon\, C^{N+1} \mid x \in H_0, |\varepsilon| < \delta(x)/_M\}$, then, by the remark above, G_0 is an open subset of C^{N+1} and for each t, $t \in E^1$, φ is analytic in (y, ε) over G_0.

Consider

(3.17) $$v' = \varepsilon\, m\, F(v)$$

and let $u(x, \varepsilon, t)$ and $v(x, \varepsilon, t)$ be solutions of the initial value problems $u(x, \varepsilon, 0) = v(x, \varepsilon, 0) = x$ for equations (3.11) and (3.17) respectively. We will show that

(3.18) $$u(x, \varepsilon, t) = \varphi(v(x, \varepsilon, t), \varepsilon, t).$$

First, it is easily seen that

$$u(x, \varepsilon, t) = w(x, \varepsilon \int_0^t f(\tau)\, d\tau) ,$$

$$v(x, \varepsilon, t) = w(x, \varepsilon\, m\, t).$$

Consequently, for $(x,\varepsilon) \in G_0$ and

$$|\varepsilon mt| < \delta(x) , \quad |\varepsilon \int_0^t f(\tau) d\tau| < \delta(x) ,$$

we have, from (3.14),

$$\varphi(v(x, \varepsilon, t), \varepsilon, t) = w(v(x, \varepsilon, t), \varepsilon\, a(t))$$

$$= w(w(x, \varepsilon mt), \varepsilon\, a(t)) = w(x, \varepsilon \int_0^t f(\tau)\, d\tau)$$

$$= u(x, \varepsilon, t),$$

which proves (3.18).

The relations $g_1(y) = m\, F(y)$, $g_n(y) = 0$ $(n > 1)$ and $\varphi_n(y, t) = a(t)^n H_n(y)$ must follow from (2.27) and (2.28). However, in this case, it is a more difficult route.

If $m = 0$, then all solutions $u(x, \varepsilon, t)$ of (3.11) with $(x, \varepsilon) \in G_0$ are periodic of period T. However, solutions with $(x, \varepsilon) \in G$ and (x, ε) a boundary point of G_0 need not be periodic. For example, consider the simple scalar equation

$$u' = \varepsilon(\sin t)\, u^2$$

whose solutions are

$$u(x, \varepsilon, t) = \frac{x}{1-\varepsilon x\, (1-\cos t)} .$$

Here, $G = C^2$, $M = 2$, $m = 0$, $\delta(x) = 1/|x|$, and consequently $G_0 = \{(x, \varepsilon) \in C^2 \mid |x\varepsilon| < 1/2\}$. The real solutions with $x\,\varepsilon \geq 1/2$ go to infinity in finite time.

Finally, for our third class of equations we will restrict the discussion to real variables. Consider

$$(3.19) \qquad u' = \varepsilon\, f(u, t)\, u$$

where $f(u, t)$ is a scalar function continuous in (u, t) and periodic in t of period $T > 0$. Also for all t, $f(u, t)$ is analytic in u over E^N and also homogeneous of degree $\mu \neq 0$ in u, i.e. $f(\lambda u, t) = \lambda^\mu f(u, t)$.

We define the scalar function $\alpha(x, \varepsilon, t)$, with $\alpha(x, \varepsilon, 0) = 1$, such that

$$(3.20) \qquad u(x, \varepsilon, t) = \alpha(x, \varepsilon, t)x$$

This leads to the scalar equation

$$\alpha' = \varepsilon\, \alpha^{\mu+1} f(x, t)$$

and consequently

$$(3.21) \qquad \alpha(x, \varepsilon, t) = \frac{1}{\left[1-\mu\varepsilon \int_0^t f(x, \tau)\, d\tau\right]^{1/\mu}}$$

Now let

$$b(x) = \frac{1}{T} \int_0^T f(x, \tau) d\tau,$$

then $b(\lambda x) = \lambda^\mu b(x)$ and the equation

$$(3.22) \qquad v' = \varepsilon\, b(v)\, v$$

is of the same type as (3.19) and therefore

$$(3.23) \qquad v(x, \varepsilon, t) = \frac{1}{[1-\mu\ \varepsilon b(x)t]^{1/\mu}}\ x$$

We observe that $u(x, \varepsilon, T) = v(x, \varepsilon, T)$. Now define

$$\varphi(y, \varepsilon, t) = u(v(y, \varepsilon, -t), \varepsilon, t)$$

then using the homogeneity of $f(u, t)$ and (3.20), (3.21) and (3.23) we find

$$\varphi(y, \varepsilon, t) = \frac{1}{[1 + \mu\varepsilon q(y,t)]^{1/\mu}}\ y$$

where

$$q(y, t) = b(y)\, t - \int_0^t f(y, \tau)\, d\tau,$$

$$q(y, t+T) = q(y, t).$$

Also, in the same way, one verifies that

$$u(x, \varepsilon, t) = \varphi(v(x, \varepsilon, t), \varepsilon, t).$$

If G is an open subset of E^N with compact closure $\bar{G}$ and

$$M = \sup |q(x, t)|, \ x \in \bar{G},\ 0 \leq t \leq T,$$

then $\varphi(y, \varepsilon, t)$, for each t, is analytic in (y, ε) for $y \in G$, $|\varepsilon| < 1/M|\mu|$.

Let $h(x, \varepsilon) = v(x, \varepsilon, T)$ where $v(x, \varepsilon, t)$ is given by (3.23), then if we expand

$$h(x, \varepsilon) = x + \sum_{n=1}^{\infty} \varepsilon^n h_n(x)$$

we find $h_1(x) = T\, b(x)x$. Using the homogeneity of $b(x)$, one may verify that

$$\frac{\partial h(x, \varepsilon)}{\partial \varepsilon} = h_1(h(x, \varepsilon)). \tag{3.24}$$

One may also verify that (3.24) is valid for the class (3.11). In Section 4 we will show that (3.24) is a necessary and sufficient condition for the function $g(y, \varepsilon)$ in the stroboscopic equation

$$v' = g(v, \varepsilon)$$

to be linear in ε.

4. GENERAL THEORY

Consider the initial value problems

$$(4.1) \qquad u' = f(u, t) \ , \ u(x, 0) = x \ ,$$

$$(4.2) \qquad v' = g(v) \qquad , \ v(x, 0) = x \ .$$

where $f(x, t)$ is a continuous function on E^{N+1} into E^N and $g(x)$ is a continuous function on E^N into E^N. If $f(u, t)$ is periodic in t of period $T > 0$, then Urabe [3] has shown that the existence of an autonomous system (4.2) whose solutions satisfy $v(x, T) = u(x, T)$ implies the existence of a function $\varphi(x, t)$ which relates the solutions according to

$$u(x, t) = \varphi(v(x, t), t)$$

and which is periodic in t with period T. We shall repeat part of Urabe's proof here with some modifications.

It is required that the solutions of (4.1) and (4.2) be unique and exist for all t. These hypotheses, which are imposed here for the sake of simplicity, yield a global result. They can be modified to produce a local result. As is well known, $v(x, t)$ is a dynamical system; in particular, $v(v(x, t),\tau) = v(x, t+\tau)$ for all $x \in E^N$, $t, \tau \in E^1$. Suppose there exists a continuous function $\varphi(x, t)$ on E^{N+1} into E^N such that

$$(4.3) \qquad u(x, t) = \varphi(v(x, t), t)$$

Setting $x = v(y, -t)$ we obtain

$$(4.4) \qquad \varphi(y, t) = u(v(y, -t), t)$$

Conversely, if $\varphi(y, t)$ is defined by (4.4), then (4.3) follows by direct substitution. Hence there exists a unique function $\varphi(y, t)$ which satisfies (4.3).

If $f(x, t)$ is periodic in t with period T, and if $v(x, T) = u(x, T)$, then $u(u(x, nT),t) = u(x, nT + t)$ and $u(x, nT) = v(x, nT)$, $n = 0, \pm 1, \ldots$. Hence $v(y, -t-T) = v(v(y, -t), -T) = u(v(y, -t),-T)$ and it follows from (4.4) that $\varphi(y, t+T) = u(v(y, -t), -T + t + T) = \varphi(y, t)$. Therefore $\varphi(y, t)$ is periodic in t with period T. Finally, if $f(y, \varepsilon, t)$ and $g(y, \varepsilon)$ are analytic in (y, ε), then so are the solutions $u(x, \varepsilon, t)$ and $v(x, \varepsilon, t)$ of (4.1) and (4.2) and hence, $\varphi(y, \varepsilon, t)$, defined by (4.4) is analytic in (y, ε).

Thus, in order to show existence of a stroboscopic representation as defined in Section 2, it suffices to show the existence of an autonomous system (2.12) whose solutions satisfy $v(x, \varepsilon, T) = u(x, \varepsilon, T)$. Therefore, one approach to the problem is to seek hypotheses on $u(x, \varepsilon, T)$ which imply the existence of a corresponding autonomous system, and then attempt to find conditions on $f(u, \varepsilon, t)$ which imply the hypotheses imposed on $u(x, \varepsilon, T)$. So far as the first part of this method is concerned, $u(x, \varepsilon, T)$ can be replaced by a given transformation $h(x, \varepsilon)$ and the problem can be formulated as follows. Given an analytic function $h(x, \varepsilon)$ which has a local expansion of the form (more precise hypotheses to be given later)

$$(4.5) \qquad h(x, \varepsilon) = x + \sum_{n=1}^{\infty} \varepsilon^n h_n(x) \quad ,$$

and given $T > 0$, find an analytic function $g(x, \varepsilon)$ of the form

$$(4.6) \qquad g(x, \varepsilon) = \sum_{n=1}^{\infty} \varepsilon^n g_n(x)$$

such that the solution $v(x, \varepsilon, t)$ of the initial value problem

$$(4.7) \qquad \begin{aligned} &(a) \quad v' = g(v, \varepsilon) \\ &(b) \quad v(x, \varepsilon, 0) = x \end{aligned}$$

satisfies the relation

$$(4.8) \qquad v(x, \varepsilon, T) = h(x, \varepsilon)$$

This is the problem to be discussed in the present section and in Section 5. This problem has been found to be of interest in itself by others. Sharshanov [4], [5], and Urabe [3] have studied the problem and produced sufficient conditions for the existence of $g(x, \varepsilon)$. (In their work the parameter ε does not appear, but the problem discussed here can be put in their setting by appropriate modifications.) However, their conditions exclude functions $h(x, \varepsilon)$ of the form (4.5), i.e., small perturbation of the identity.

Part of the discussion in this section is formal. In particular, in certain instances power series are manipulated without justification. Certain rigorous results are summarized as lemmas. Throughout this section it will be assumed that $h(x, \varepsilon)$ is an analytic function of (x, ε) on G into E^N where G is a connected open subset of E^{N+1} such that $E^N \times \{0\} \subset G$. Moreover, it will be assumed that $h(x, 0) = x$ for all $x \in E^N$. It follows that given a compact set $A \subset E^N$, there exists $\delta > 0$ such that $A \times [-\delta, \delta] \subset G$ and such that the series (4.5) converges uniformly in (x, ε) on $A \times [-\delta, \delta]$. The functions $h_n(x)$ are single valued analytic functions on E^N into E^N.

Proceeding formally, we seek an analytic function $g(x, \varepsilon)$ of the form (4.6) such that the solution of (4.7) satisfies (4.8). Since $g(x, 0) = 0$, $v'(x, 0, t) = 0$ and hence $v(x, 0, t) = x$. Thus $v(x, \varepsilon, t)$ has an expansion of the form

$$(4.9) \qquad v(x, \varepsilon, t) = x + \sum_{n=1}^{\infty} \varepsilon^n v_n (x, t)$$

where $v_n(x, 0) = 0$, $n = 1, 2, \ldots$. Referring back to Section 2, v has the same form as φ, and g has the same form as f for fixed t. Therefore, the expansion of $g(v(x, \varepsilon, t), \varepsilon)$ is directly obtainable from (2.24):

$$\text{(a)}\quad g(v(x, \varepsilon, t), \varepsilon) = \sum_{n=1}^{\infty} \varepsilon^n g_n(x) + \sum_{n=2}^{\infty} \varepsilon^n \sum_{\nu=1}^{n-1} V_{n-\nu,\nu}(x, t)$$

(4.10)

$$\text{(b)}\quad V_{m\nu}(x, t) = \sum_{k=1}^{\nu} \frac{1}{k!} \sum_{j_1+\cdots+j_k=\nu} \sum_{i_1,\ldots,i_k=1}^{N} v_{j_1}^{(i_1)}(x,t)\cdots v_{j_k}^{(i_k)}(x,t) \frac{\partial^k g_m(x)}{\partial x^{(i_1)}\cdots\partial x^{(i_k)}}$$

Substituting (4.9) and (4.10) into (4.7) and equating coefficients of like powers of ε, we obtain

$$v_1'(x, t) = g_1(x)$$

(4.11)

$$v_n'(x, t) = g_n(x) + \sum_{\nu=1}^{n-1} V_{n-\nu,\nu}(x, t)\ , \quad n = 2, 3, \ldots$$

For $\nu = 1, \ldots, n-1$, the functions $V_{n-\nu,\nu}(x, t)$ are expressed in terms of $v_1, \ldots, v_{n-1}$ and $g_1, \ldots, g_{n-1}$. From (4.7b), $v_n(x, 0) = 0$, $n = 1, 2, \ldots$. Hence (4.11) uniquely determines the v_n in terms of the g_n:

$$v_1(x, t) = t\, g_1(x)$$

(4.12)

$$v_n(x, t) = t\, g_n(x) + \sum_{\nu=1}^{n-1} \int_0^t V_{n-\nu,\nu}(x, \tau)\, d\tau, \quad n = 2, 3, \ldots$$

Then the g_n are uniquely determined by (4.8) and (4.12):

$$(4.13)\qquad \begin{aligned} g_1(x) &= \frac{1}{T} h_1(x) \\ g_n(x) &= \frac{1}{T} h_n(x) - \frac{1}{T} \sum_{\nu=1}^{n-1} \int_0^T V_{n-\nu,\nu}(x, \tau)\, d\tau, \quad n = 2, 3, \ldots \end{aligned}$$

This establishes uniqueness of the function $g(x, \varepsilon)$.

It may be noted that the requirement $g(x, 0) = 0$ is essential to the uniqueness result. For example, let A be a non-zero real constant matrix such that $e^{TA} = I$ and let $h(x, \varepsilon) = x$ for all $x \in E^N$. Then $g(x, \varepsilon) = 0$ satisfies the above requirements and so does $g(x, \varepsilon) = Ax$ except for the condition $g(x, 0) = 0$. The requirement of analyticity is also essential to uniqueness as is shown by the following one-dimensional example. Let $T = 2\pi$ and $h(x, \varepsilon) = x + 2\pi\varepsilon$. Then $g(x, \varepsilon) = \varepsilon$ is the unique analytic function which satisfies the above conditions. Given any positive integer n, let

$$g(x, \varepsilon) = \frac{\varepsilon}{1 + \varepsilon^n \cos \frac{x}{\varepsilon}}$$

for $|\varepsilon| < 1$ (with $g(x, 0) = 0$). This function has continuous derivatives of order $[\frac{n-1}{2}]$ with respect to x and ε. Moreover, it is single valued and analytic in both x and ε if $0 < |\varepsilon| < 1$. The solution of the differential equation

$$v' = \frac{\varepsilon}{1 + \varepsilon^n \cos \frac{v}{\varepsilon}}$$

with initial value x is uniquely determined by the equation

$$v - x + \varepsilon^{n+1} [\sin \frac{v}{\varepsilon} - \sin \frac{x}{\varepsilon}] = \varepsilon t$$

which is obtained by integration. Since $v = x + 2\pi\varepsilon$ satisfies this equation with $t = 2\pi$, we conclude that $v(x, \varepsilon, 2\pi) = x + 2\pi\varepsilon = h(x, \varepsilon)$. Thus smoothness requirements on $g(x, \varepsilon)$ short of analyticity are not sufficient to give uniqueness.

If (4.6) converges to an analytic function $g(x, \varepsilon)$, then by known existence theorems, the solution $v(x, \varepsilon, t)$ of (4.7) is analytic in (x, ε). Since $g(x, 0) = 0$, it follows that given any compact set $A \subset E^N$, there exists $\delta > 0$ such that the solution $v(x, \varepsilon, t)$ exists for $0 \leq t \leq T$ if $x \in A$ and $|\varepsilon| < \delta$. Then, for $(x, \varepsilon) \in A \times (-\delta, \delta)$, the formal manipulations which produced (4.12) and (4.13) are justified and it follows that $v(x, \varepsilon, T) = h(x, \varepsilon)$. However, attempts to find sufficient conditions for convergence with the use of (4.10b), (4.12), and (4.13) have been unsuccessful (except for the case where $h(x, \varepsilon)$ is linear, a case which was dealt with more easily in Section 3). If all but a finite number of the functions $g_n(x)$ are 0 for all $x \in E^N$, then obviously (4.6) defines an analytic function on all of E^{N+1}. We now present a simple characterization of a very special class of functions $h(x, \varepsilon)$, those for which $g_n(x) = 0$ for $n \geq 2$.

<u>Lemma 4.1</u>. Let $h(x, \varepsilon)$ be analytic on G into E^N, where G is a connected open set in E^{N+1} such that $E^N \times \{0\} \subset G$, and assume $h(x, 0) = x$ for all $x \in E^N$ whereby $h(x, \varepsilon)$ has a local expansion of the form (4.5). Let the functions $g_n(x)$ be defined by (4.10b), (4.12) and (4.13). Then $g_n(x) = 0$ for $n \geq 2$, $x \in E^N$, if and only if

$$(*) \qquad \frac{\partial}{\partial\varepsilon} h(x, \varepsilon) = h_1(h(x, \varepsilon))$$

for all $(x, \varepsilon) \in G$.

Proof: Given (*) and given any bounded open set $U \subset E^N$, choose $\delta > 0$ such that $U \times (-\delta, \delta) \subset G$. Then the function $v(x, \varepsilon, t) = h(x, \frac{\varepsilon}{T} t)$ is defined for $0 \leq t \leq T$, $(x, \varepsilon) \in U \times (-\delta, \delta)$, $v(x, \varepsilon, 0) = x$, $v(x, \varepsilon, T) = h(x, \varepsilon)$, and $v(x, \varepsilon, t)$ satisfies the differential equation

$$v' = \frac{\varepsilon}{T} h_1(v)$$

Hence $g(x, \varepsilon) = \frac{\varepsilon}{T} h_1(x)$ satisfies the conditions previously imposed and so the coefficients in its expansion in powers of ε are uniquely determined by (4.12) and (4.13). Therefore $g_1(x) = \frac{1}{T} h_1(x)$ and $g_n(x) = 0$ for $n \geq 2$, $x \in U$. Since U is arbitrary, the result follows for all $x \in E^N$.

Conversely, if $g_n(x) = 0$ for $n \geq 2$, $x \in E^N$, then the series (4.6) converges to the analytic function $g(x, \varepsilon) = \varepsilon\, g_1(x)$. Given any bounded open set $U \subset E^N$, there exists $\delta > 0$ such that $U \times (-\delta, \delta) \subset G$ and such that the solution $v(x, \varepsilon, t)$ of (4.7) is defined for $0 \leq t \leq T$ for $(x, \varepsilon) \in U \times (-\delta, \delta)$. It then follows from (4.12) and (4.13) that $v(x, \varepsilon, T) = h(x, \varepsilon)$. Let $w(x, t)$ be the solution of the initial value problem

$$(**) \qquad \begin{aligned} w' &= g_1(w) \\ w(x, 0) &= x \end{aligned}$$

Then $v(x, \varepsilon, t) = w(x, \varepsilon t)$. Hence $h(x, \varepsilon) = w(x, \varepsilon T)$ and substitution into (**) yields

$$\frac{\partial}{\partial \varepsilon} h(x, \varepsilon) = T\, g_1(h(x, \varepsilon))$$

Setting $\varepsilon = 0$ we see that $h_1(x) = T\, g_1(x)$ and therefore (*) holds on $U \times (-\delta, \delta)$. But both sides of (*) are analytic on all of G and hence (*) holds on G (e.g., see [11], pp. 34 and 35). This completes the proof.

In connection with lemma 4.1 we note that although $g(x, \varepsilon) = \varepsilon g_1(x)$ is analytic on all of E^{N+1} if $g_n(x) = 0$ for $n \geq 2$, $h(x, \varepsilon)$ may not be since the solution of (4.7) may not exist for all $t \in [0, T]$. For example, let $N = 1$ and $g(x, \varepsilon) = \varepsilon x^2$. Then

$$v(x, \varepsilon, t) = \frac{x}{1 - \varepsilon x t}$$

and

$$h(x, \varepsilon) = \frac{x}{1-\varepsilon x T} .$$

Hence

$$G = \{(x, \varepsilon) | \ \varepsilon x < T\} .$$

We shall now pursue another approach to the problem wherein the solution $v(x, \varepsilon, t)$ of (4.7) is expressed in terms of the iterations of $h(x, \varepsilon)$ without simultaneously constructing the function $g(x, \varepsilon)$. In addition to the hypotheses previously imposed on $h(x, \varepsilon)$ it will be assumed that $(h(x, \varepsilon), \varepsilon) \in G$ for all $(x, \varepsilon) \in G$. This implies that the iterations of $h(x, \varepsilon)$, defined inductively by the relations

$$\begin{aligned} &h(x, \varepsilon, 1) = h(x, \varepsilon) \\ (4.14) \qquad & \\ &h(x, \varepsilon, j+1) = h(h(x, \varepsilon, j), \varepsilon), j=1, 2, \dots , \end{aligned}$$

are analytic in (x, ε) on G. It follows that

$$(4.15) \qquad h(h(x, \varepsilon, i), \varepsilon, j) = h(x, \varepsilon, i+j), \ i, j=1, 2, \dots$$

With the use of (2.23), with $\varphi(x, \varepsilon)$ replaced by $h(x, \varepsilon, j)$, it can be shown that the series expansion of $h(x, \varepsilon, j+1)$, in powers of ε has the form

$$(4.16) \qquad h(x, \varepsilon, j+1) = x + \sum_{n=1}^{\infty} \varepsilon^n h_n(x,j+1), \ j=1, 2, \dots$$

where

$$h_n(x, 1) = h_n(x)$$

$$h_1(x, j+1) = h_1(x) + h_1(x, j) \, , \quad j = 1, 2, \ldots$$

(4.17) $$h_n(x, j+1) = h_n(x) + h_n(x, j) + H_n(x, j), \quad n = 2, 3, \ldots$$

$$H_n(x, j) = \sum_{\nu=1}^{n-1} \sum_{k=1}^{\nu} \frac{1}{k!} \sum_{j_1+\cdots+j_k=\nu} \sum_{i_1,\ldots,i_k=1}^{N} h_{j_1}^{(i_1)}(x,j)\cdots h_{j_k}^{(i_k)}(x, j).$$

$$\frac{\partial^k h_{n-\nu}(x)}{\partial x^{(i_1)}\cdots\partial x^{(i_k)}}$$

It also follows from (2.23), with $\ell(x, \varepsilon)$ replaced by $h(x, \varepsilon, i)$ and $h(x, \varepsilon)$ replaced by $h(x, \varepsilon, j)$ that

(4.18)
$$h(h(x, \varepsilon, i), \varepsilon, j) = x + \sum_{n=1}^{\infty} \varepsilon^n \, [h_n(x, i) + h_n(x, j)] + \sum_{n=2}^{\infty} \varepsilon^n H_n(x, i, j) \, , \quad i, j=1, 2, \ldots$$

where

(4.19)
$$H_n(x, i, j) = \sum_{\nu=1}^{n-1} \sum_{k=1}^{\nu} \frac{1}{k!} \sum_{j_1+\cdots+j_k=\nu} \sum_{i_1,\ldots,i_k=1}^{N} h_{j_1}^{(i_1)}(x,i)\cdots h_{j_k}^{(i_k)}(x, i).$$

$$\frac{\partial^k h_{n-\nu}(x, j)}{\partial x^{(i_1)}\cdots\partial x^{(i_k)}}$$

From (4.15), (4.16), and (4.18) we obtain the relations

$$(4.20) \quad \begin{aligned} &(a)\ h_1(x,i) + h_1(x,j) = h_1(x, i+j) \\ &(b)\ h_n(x,i) + h_n(x,j) + H_n(x,i,j) = h_n(x,i+j),\ n= 2,3,\dots, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad i, j=1, 2, \dots . \end{aligned}$$

These relations are rigorous consequences of the hypotheses imposed on $h(x, \varepsilon)$. For by hypothesis and known theory of analytic functions (e.g., see [11], p. 35), the functions $h(x, \varepsilon, j)$ are analytic on G for all $j = 1, 2, \dots$. Given a bounded open set $U \subset E^N$ and given positive integers i and j, there exists $\delta > 0$ (which, in general, may depend on U, i, and j) such that the relations (4.16) through (4.19) are valid for $x \in U$ and $|\varepsilon| < \delta$. The number δ may tend to 0 as $i \to \infty$ or $j \to \infty$, but this does not effect the result since the relations (4.17) and (4.20) are independent of ε .

Actually, the hypothesis $(h(x, \varepsilon), \varepsilon) \in G$ for all $(x, \varepsilon) \in G$ is redundant so far as (4.20) is concerned. Because $h(x, \varepsilon)$ is a small perturbation, given a bounded open set U and a positive integer i, there exists $\delta > 0$ such that $(h\ (x, \varepsilon, j), \varepsilon) \in G$ for $j = 1, \dots, i$, $x \in U$, and $|\varepsilon| < \delta$, and the above arguments go through. In fact if the $h_n(x)$ are arbitrary analytic functions on E^N into E^N, without regard for convergence of (4.5), and if the $h_n(x, j)$ are defined by (4.17), it seems likely that (4.20) can be established by a similar argument with the use of truncations of (4.5).

It follows from the first two equations of (4.17) that $h_1(x, j) = j\, h_1(x)$, $j = 1, 2, \dots$. Thus, defining $p_1(x, t)$ to be

$$(4.21) \qquad p_1(x, t) = t\, h_1(x)$$

we have $h_1(x, j) = p_1(x, j)$, $j = 1, 2, \dots$. It will now be shown by induction that for each $m = 1, 2, \dots$ there exists a function $p_m(x, t)$

which is a polynomial in t of degree $\leq m$ whose coefficients are analytic functions on E^N into E^N such that

$$p_m(x, j) = h_m(x, j), \; j = 1, 2, \ldots \tag{4.22}$$

Assume it true for $m \leq n - 1$ and let

$$Q_n(x,t) = \sum_{\nu=1}^{n-1} \sum_{k=1}^{\nu} \frac{1}{k!} \sum_{j_1+\cdots+j_k=\nu} \quad \sum_{i_1,\ldots,i_k=1}^{N} p_{j_1}^{(i_1)}(x,t)\cdots p_{j_k}^{(i_k)}(x,t) \cdot \frac{\partial^k h_{n-\nu}(x)}{\partial x^{(i_1)}\cdots\partial x^{(i_k)}} \tag{4.23}$$

It follows from (4.17) and the induction assumption that $H_n(x,j) = Q_n(x, j)$, $j = 1, 2, \ldots$, and hence, by (4.17),

$$h_n(x, j + 1) - h_n(x, j) = h_n(x) + Q_n(x, j) \tag{4.24}$$

By induction assumption and (4.23), $h_n(x) + Q_n(x, t)$ is a polynomial in t of degree $\leq n - 1$ and therefore the nth difference of the right side of (4.24) with respect to j is 0. This implies that the $(n + 1)$th difference of $h(x, j)$ with respect to j is 0, and hence for each fixed x there exists a polynomial $p_n(x, t)$ in t of degree $\leq n$ which satisfies (4.22). Moreover, from (4.24), $p_n(x, j+1)-p_n(x, j) = h_n(x) + Q_n(x, j)$, $j = 1, 2, \ldots$ and this implies

$$p_n(x, t + 1) - p_n(x, t) = h_n(x) + Q_n(x, t), \; n = 2, 3, \ldots \tag{4.25}$$

for all real t. We now show by induction that $p_n(x, 0) = 0$, $n = 1, 2, \ldots$. By (4.21) it holds for $m=1$. If it is true for $m \leq n - 1$, then by (4.23), $Q_n(x,0) = 0$, and by (4.25), $p_n(x, 1) - p_n(x, 0) = h_n(x)$. But $p_n(x, 1) = h_n(x, 1) = h_n(x)$ and hence $p_n(x, 0) = 0$.

The polynomial $p_n(x, t)$ is uniquely determined by the $n + 1$ values $p_n(x, 0) = 0$, $p_n(x, j) = h_n(x, j)$, $j = 1, \ldots, n$, and it is easily seen that

$$(4.26) \qquad p_n(x, t) = \frac{1}{\Delta_n} \sum_{i=1}^{n} \Delta_{ni}(t)\, h_n(x, i)$$

where

$$(4.27) \qquad \Delta_n = \begin{vmatrix} 1 & 2 & \cdot & n \\ 1 & 2^2 & \cdot & n^2 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 2^n & \cdot & n^n \end{vmatrix}, \quad \Delta_{ni} = \begin{array}{c} \text{ith Column} \\ \begin{vmatrix} 1 & \cdots & t & & n \\ 1 & \cdots & t^2 & & n^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \cdots & t^n & & n^n \end{vmatrix} \end{array}$$

This shows that the coefficients of $p_n(x, t)$ are analytic functions of x on E^N. The following lemma has now been established.

<u>Lemma 4.2</u> Let G be a connected open set in E^{N+1} such that $E^N \times \{0\} \subset G$. Let $h(x, \varepsilon)$ be an analytic function of (x, ε) on G into E^N such that $h(x, 0) = x$ and $(h(x, \varepsilon), \varepsilon) \in G$ for all $(x, \varepsilon) \in G$. Then the iterations $h(x, \varepsilon, j)$, $j = 1, 2, \ldots$, defined by (4.15), are analytic on G and have local expansions of the form (4.16) where the $h_n(x, j)$ are given by (4.17) and satisfy (4.20). The function $p_n(x, t)$ defined by (4.26) is a polynomial in t of degree $\leq n$, $p_n(x, 0) = 0$, and $p_n(x, j) = h_n(x, j)$, $j = 1, 2, \ldots$.

Proceeding formally, we now let

$$(4.28) \qquad p(x, \varepsilon, t) = x + \sum_{n=1}^{\infty} \varepsilon^n p_n(x, t)$$

Then $p(x, \varepsilon, 0) = x$ and $p(x, \varepsilon, 1) = h(x, \varepsilon)$. Thus $p(x, \varepsilon, t)$ satisfies (4.8) with $T = 1$. It will now be shown that (4.28) formally defines a dynamical system, i.e.,

$$(4.29) \qquad p(p(x, \varepsilon, t), \varepsilon, \tau) = p(x, \varepsilon, t + \tau) .$$

From (2.23),

$$p(p(x, \varepsilon, t), \varepsilon, \tau) = x + \sum_{n=1}^{\infty} \varepsilon^n [p_n(x, t) + p_n(x, \tau)] \tag{4.30}$$
$$+ \sum_{n=2}^{\infty} \varepsilon^n Q_n(x, t, \tau)$$

where

$$Q_n(x,t,\tau) = \sum_{\nu=1}^{n-1} \sum_{k=1}^{\nu} \frac{1}{k!} \sum_{j_1+\dots+j_k=\nu} \sum_{i_1,\dots,i_k=1}^{N} p_{j_1}^{(i_1)}(x,t)\dots p_{j_k}^{(i_k)}(x,t). \frac{\partial^k p_{n-\nu}(x, \tau)}{\partial x^{(i_1)}\dots\partial x^{(i_k)}} \tag{4.31}$$

Hence (4.29) is satisfied if and only if

$$p_1(x, t) + p_1(x, \tau) = p_1(x, t + \tau)$$
$$(4.32)$$
$$p_n(x, t) + p_n(x, \tau) + Q_n(x, t, \tau) = p_n(x, t + \tau), \; n = 2, 3, \dots$$

It will now be shown , independently of the question of convergence of (4.28) and (4.30), that (4.32) is satisfied.

<u>Lemma 4.3</u> If the hypothesis of lemma 4.2 is fulfilled, then the functions $p_n(x, t)$ defined by (4.26) satisfy (4.32).

Proof: By lemma 4.2, $p_n(x, j) = h_n(x, j)$, $j = 1, 2, \dots$, and it follows from (4.19) and (4.31) that $Q_n(x, i, j) = H_n(x, i, j)$, $i, j = 1, 2, \dots$. It then follows from (4.20) that (4.32) is satisfied for $(\tau, t) = (i, j)$, $i, j = 1, 2, \dots$. Given any positive integer i, if $t = i$ the relations (4.32) become polynomial equations in τ which are satisfied for $\tau = j$, $j = 1, 2, \ldots$. Therefore they are satisfied for all real τ. Thus (4.32) holds for all real τ if t is a positive integer. For fixed τ, these relations are polynomial equations in t which hold for $t = i$, $i = 1, 2, \ldots$ and therefore for all real t.

It now follows that if (4.28) converges to an analytic function and if the manipulations which produced (4.30), (4.31) and (4.32) are justifiable, then $p(x, \varepsilon, 0) = x$, $p(x, \varepsilon, 1) = h(x, \varepsilon)$, and (4.29) is satisfied. (These remarks contain tacit assumptions as to the region of convergence and, in particular, analyticity of $p(x, \varepsilon, t)$ in (x, ε) at least for $0 \le t \le 1$.) Then $v(x, \varepsilon, t) = p(x, \varepsilon, \frac{t}{T})$ satisfies (4.7b), (4.8), and has the group property (4.29). From this it can be shown (see Section 5) that $v(x, \varepsilon, t)$ satisfies (4.7a) with $g(x, \varepsilon) = v'(x, \varepsilon, 0)$. Thus an investigation of the convergence of (4.28) affords us an alternate method of dealing with the problem formulated at the beginning of this section. This method will be used in Section 5 to prove a convergence theorem.

5. A CONVERGENCE THEOREM

Let

$$(5.1) \qquad \theta_{ni}(t) = \frac{\Delta_{ni}(t)}{\Delta_n}, \quad i = 1, \ldots, n, \quad n = 1, 2, \ldots .$$

where $\Delta_{ni}(t)$ and Δ_n are given by (4.27). It follows from (4.27) that $\theta_{ni}(j) = 0$, $j \neq i$, $j = 0, \ldots, n$, and $\theta_{ni}(i) = 1$. From this it is easily seen that (with an obvious modification of notation for $i = n$)

$$(5.2) \qquad \theta_{ni}(t) = (-1)^{n-i} \cdot \frac{t(t-1) \ldots (t-i+1)(t-i-1) \ldots (t-n)}{i!\,(n-i)!},$$

$$i = 1, \ldots, n$$

$$n = 1, 2, \ldots$$

Differentiation yields

$$(5.3) \qquad \theta'_{ni}(t) = \frac{(-1)^{n-i}}{i!(n-i)!} \sum_{\substack{j=0 \\ j\neq i}}^{n} \prod_{\substack{\ell=0 \\ \ell\neq i \\ \ell\neq j}}^{n} (t-\ell)$$

We begin by finding upper bounds for $|\theta_{ni}(t)|$ and $|\theta'_{ni}(t)|$. For this purpose use will be made of the following lemma.

<u>Lemma 5.1</u> Given a positive integer n and any $t \in [0, n]$, let $j_0, j_1, \ldots, j_n$ be the integers in $[0, n]$ enumerated in order of their distances from t, i.e., $|t-j_k| \leq |t-j_{k+1}|$, $k = 0, \ldots, n-1$. Then $|t-j_0| \leq 1$ and $|t-j_k| \leq k$, $k = 1, \ldots, n$.

Proof: Clearly $|t-j_{k+1}| \leq |t-j_k| + 1$ and therefore the result follows by induction on k once it has been established that $|t-j_0| \leq 1$ and $|t-j_1| \leq 1$. If t is an integer then $j_0 = t$ and hence $|t-j_0| = 0$, $|t-j_1| = 1$. Otherwise t is interior to a unit interval with j_0 and j_1 as end points. Hence $|t-j_0| < 1$ and $|t-j_1| < 1$.

Given a positive integer n, consider the function $\theta_{ni}(t)$ for any $i = 1, \ldots, n$. Given $t \in [0, n]$ let $j_0, j_1, \ldots, j_n$ be the quantities defined in lemma 5.1. One of these integers is necessarily i, i.e., there exists $k_0 \in \{0, 1, \ldots, n\}$ such that $j_{k_0} = i$. It follows with the use of lemma 5.1 that

$$\prod_{\substack{j=0 \\ j\neq i}}^{n} |t-j| = \prod_{\substack{k=0 \\ k\neq k_0}}^{n} |t-j_k| \leq \prod_{\substack{k=1 \\ k\neq k_0}}^{n} k \leq n!$$

and therefore, by (5.2),

$$|\theta_{ni}(t)| \leq \binom{n}{i} , \quad 0 \leq t \leq n \tag{5.4}$$

From (5.3)

$$|\theta'_{ni}(t)| \leq \frac{1}{i!(n-i)!} \sum_{\substack{j=0 \\ j\neq i}}^{n} \prod_{\substack{\ell=0 \\ \ell\neq i \\ \ell\neq j}}^{n} |t-\ell|$$

It follows by the same reasoning used above that each of the n products on the right side of this inequality is bounded by $n!$ and hence

$$|\theta'_{ni}(t)| \leq n\binom{n}{i} , \quad 0 \leq t \leq n \tag{5.5}$$

The inequality $\binom{n}{i} < 2^n$ follows from the relation $\sum_{i=0}^{n} \binom{n}{i} = 2^n$, and hence (5.4) and (5.5) yield

$$(5.6) \qquad \begin{array}{l} |\theta_{ni}(t)| < 2^n \\ \\ |\theta'_{ni}(t)| < n\, 2^n \end{array} \quad , \quad 0 \leq t \leq n, \quad i = 1, \ldots, n$$

Given any positive integer m, it is clear that $|\theta_{ni}(t)|$ assumes its maximum value on the interval $-m \leq t \leq 0$ at $t = -m$ $(i=1, \ldots, n)$. Hence for $-m \leq t \leq 0$,

$$|\theta_{ni}(t)| \leq \frac{m(m+1)\ldots(m+n)}{i!\,(n-i)!} = m\binom{m+n}{m}\binom{n}{i} < m.2^m.2^{2n}$$

A similar bound is obtainable for $|\theta'_{n_i}(t)|$, and since these bounds exceed the ones in (5.6) we have

$$(5.7) \qquad \begin{array}{l} |\theta_{ni}(t)| < m\, 2^m\, 2^{2n} \\ \\ |\theta'_{ni}(t)| < m2^m\, n2^{2n} \end{array} \quad , \quad -m \leq t \leq n, \; i = 1, \ldots, n$$

As in Section 4, we are primarily interested in analytic functions $h(x, \varepsilon)$ on E^{N+1} into E^N. However, such functions can be extended analytically to an open set in C^{N+1}, and it is necessary to consider $h(x, \varepsilon)$ on a complex domain in order to invoke a theorem on uniformly convergent sequences of analytic functions. Therefore the hypotheses on $h(x, \varepsilon)$ will be stated for a complex domain. The space E^k will be regarded as imbedded in C^k, $k = N$, $N + 1$, and the norm of a vector $x \in C^N$ will be denoted by

$$\|x\| = \left(\sum_{j=1}^{N} |x^{(j)}|^2 \right)^{1/2}$$

The first two hypotheses to be imposed on $h(x, \varepsilon)$ are the following:

(i) $h(x, \varepsilon)$ is a single valued analytic function on a connected open set $V \subset C^{N+1}$ into C^N such that $E^N \times \{0\} \subset V$ and $h(x, 0) = x$ for all $x \in C^N$ for which $(x, 0) \in V$.

(ii) $(h(x, \varepsilon), \varepsilon) \in V \cap E^{N+1}$ for all $(x, \varepsilon) \in V \cap E^{N+1}$.

Let

$$(5.8) \qquad V_0 = \{x \in C^N | (x, 0) \in V\}$$

It follows from (i) that given any $x_0 \in V_0$, there exist $\rho(x_0) > 0$ and $\delta(x_0) > 0$ such that (4.5) converges uniformly in $(x, \varepsilon) \in C^{N+1}$ on the neighborhood $\|x-x_0\| < \rho(x_0)$, $|\varepsilon| < \delta(x_0)$. Moreover, on this neighborhood (4.5) can be expanded in a uniformly convergent power series in the N + 1 complex variables $x^{(1)}-x_0^{(1)}, \ldots, x^{(N)}-x_0^{(N)}, \varepsilon$. In particular, the functions $h_n(x)$ which appear in (4.5) are analytic on V_0.

It follows from (ii) that the iterates $h(x, \varepsilon, j)$, defined by (4.14), exist for all $j = 1, 2, \ldots$ if $(x, \varepsilon) \in V \cap E^{N+1}$. A$ (real) analytic functions of analytic functions they are analytic on $V \cap E^{N+1}$ (e.g., see [11, p. 35]. Thus, given $x_0 \in E^N$, there exist $\rho(x_0, i) > 0$, $\delta(x_0, i) > 0$ such that the series (4.16) converges uniformly in $(x, \varepsilon) \in E^{N+1}$ on the neighborhood $\|x-x_0\| < \rho(x_0, i)$, $|\varepsilon| < \delta(x_0, i)$, and the coefficients of the expansion are given by (4.17). Also, on this neighborhood (4.16) can be expanded in a uniformly convergent power series in the N+1 real variables $x^{(1)}-x_0^{(1)}, \ldots, x^{(N)}-x_0^{(N)}, \varepsilon$.

Without regard for the existence of the iterates $h(x, \varepsilon, j)$ for complex (x, ε), the functions $h_n(x, j)$ are defined by (4.17) as single valued analytic functions on V_0, where V_0 is defined by (5.8). The next hypothesis, which is very stringent, imposes a condition on the functions $h_n(x, i)$. It is shown at the end of this section that the class of functions which satisfy the condition is at least large enough to contain the class of linear functions properly.

(iii) Given $x_0 \in E^N$, there exists $\rho(x_0) > 0$ and $M(x_0) > 0$ such that if $x \in C^N$ and $\|x-x_0\| < \rho(x_0)$, then $x \in V_0$ and $\|h_n(x, i)\| < M(x_0)^n$ for $i = 1, \ldots, n$, $n = 1, 2, \ldots$, where the $h_n(x, i)$ are defined on V_0 by (4.17).

<u>Lemma 5.2</u> If $h(x, \varepsilon)$ satisfies hypotheses (i), (ii), and (iii), then given a bounded open set $A \subset E^N$, there exist a bounded open set $U \subset V_0$ (where V_0 is defined by (5.8)) and $M > 0$ such that $\bar{A} \subset U$ and $\|h_n(x, i)\| < M^n$, $i = 1, \ldots, n$, $n = 1, 2, \ldots$, for all $x \in U$.

Proof: Given $x \in \bar{A}$, let $s(x) = \{y \in C^N \mid \|y-x\| < \rho(x)\}$ where $\rho(x)$ is given in (iii). Since $\bar{A}$ is compact, a finite collection of the $s(x)$ cover it:

$$\bar{A} \subset \bigcup_{j=1}^{m} s(x_j)$$

Take $U = \bigcup_{j=1}^{m} s(x_j)$ and $M = \max_{j=1,\ldots,m} M(x_j)$.

With the use of the definition (5.1), (4.26) can be written

$$p_n(x, t) = \sum_{i=1}^{n} \theta_{ni}(t)\, h_n(x, i) \,. \tag{5.9}$$

It will now be shown that the series

(5.10)

$$\text{(a)} \qquad p(x, \varepsilon, t) = x + \sum_{n=1}^{\infty} \varepsilon^n p_n(x, t)$$

$$\text{(b)} \qquad p'(x, \varepsilon, t) = \sum_{n=1}^{\infty} \varepsilon^n p_n'(x, t)$$

converge to analytic functions.

Lemma 5.3 If $h(x, \varepsilon)$ satisfies hypotheses (i), (ii), (iii) then, given any bounded open set $G \subset E^N$, there exists $\delta > 0$ such that the series (5.10) converge uniformly in (x, ε, t), for $(x, \varepsilon) \in G \times (-\delta, \delta)$ and for t on any finite interval, to functions which are continuous in (x, ε, t) on $G \times (-\delta, \delta) \times E^1$ and, for fixed t, are analytic in (x, ε) on $G \times (-\delta, \delta)$. Moreover, given $x_0 \in G$, there exists $\rho(x_0) > 0$ such that the power series expansions of $p(x, \varepsilon, t)$ and $p'(x, \varepsilon, t)$ in the $N + 1$ real variables $x^{(1)}-x_0^{(1)}, \ldots, x^{(N)}-x_0^{(N)}, \varepsilon$ converge for $\|x-x_0\| < \rho(x_0)$, $|\varepsilon| < \delta$, and all real t.

Proof: By lemma 5.2 there exist a bounded open set $U \subset V_0$ and $M > 0$ such that $G \subset U$ and $\|h_n(x, i)\| < M^n$, $i = 1, \ldots, n$, $n = 1, 2, \ldots$, for all $x \in U$. Choose any δ in the interval $0 < \delta < 1/M$. Given any $T_0 > 0$, choose a positive integer $m \geq T_0$. If $t \in [-T_0, T_0]$ and $n \geq m$, then $-m \leq t \leq n$ and $|\theta_{ni}(t)|$ and $|\theta'_{ni}(t)|$ are bounded according to (5.7) It then follows from (5.9) that

$$\|p_n(x, t)\| \leq m.2^m(4M)^n n$$

$$\|p'_n(x, t)\| \leq m.2^m(4M)^n n^2$$

for all $x \in U$, $t \in [-T_0, T_0]$, $n \geq m$. Application of the ratio test shows that

$$\sum_{n=m}^{\infty} \delta^n \, m.2^m(4M)^n n < \infty$$

$$\sum_{n=m}^{\infty} \delta^n \, m.2^m(4M)^n \, n^2 < \infty$$

and these series dominate (5.10a) and (5.10b) respectively for $n \geq m$ if $(x, \varepsilon, t) \in U \times \sigma \times [-T_0, T_0]$, where $\sigma = \{\varepsilon \in C^1 \mid |\varepsilon| < \delta\}$. This establishes the uniform convergence of (5.10) on $U \times \sigma \times [-T_0, T_0]$. It follows from a theorem of Weierstrass (e.g., see [12], theorem 3.1, p. 38) that $p(x, \varepsilon, t)$ and $p'(x, \varepsilon, t)$ are analytic in (x, ε) on $U \times \sigma$ for each $t \in E^1$. Given any $x_0 \in U$ there exists $\rho(x_0) > 0$ such that $\{x \in C^N \mid \|x-x_0\| < \rho(x_0)\} \subset U$ whence $\{x \in C^N \mid \|x-x_0\| < \rho(x_0)\} \times \sigma \subset U \times \sigma$. Therefore the power series expansions of $p(x, \varepsilon, t)$ and $p'(x, \varepsilon, t)$ in the $N + 1$ complex variables $x^{(1)}-x_0^{(1)}, \ldots, x^{(N)}-x_0^{(N)}, \varepsilon$ converge for $\|x-x_0\| < \rho(x_0)$, $|\varepsilon| < \delta$ ([11], Chap. II, theorem 3, p. 33). This proves the assertions of the lemma for $(x,\varepsilon) \in U \times \sigma$ and hence, since $G \times(-\delta,\delta) \subset U \times \sigma$, they hold for $(x, \varepsilon) \in G \times \sigma$.

It follows from (4.16), (5.2), (5.9), and (5.10) that

$$
(5.11) \qquad \begin{aligned} p(x, \varepsilon, 0) &= x \\ p(x, \varepsilon, k) &= h(x, \varepsilon, k), \quad k = 1, 2, \ldots \end{aligned}
$$

for $(x, \varepsilon) \in G \times(-\delta, \delta)$.

<u>Lemma 5.4</u>. Let $h(x, \varepsilon)$ satisfy hypotheses (i), (ii), and (iii). Given a bounded connected open set $U \subset E^N$ and $T_0 > 0$, there exist a bounded connected open set $G \subset E^N$ and $\delta > 0$ for which the conclusions of lemma 5.3 hold and such that $p(x, \varepsilon, t) \in G$ for all $(x, \varepsilon, t) \in U \times(-\delta,\delta) \times [-T_0, T_0]$.

Proof: Choose any bounded open set $U_1 \subset E^N$ such that $\bar{U} \subset U_1$. By lemma 5.3, there exists $\delta_1 > 0$ such that the conclusion of lemma 5.3 holds with respect to δ_1, U_1. In particular, $p(x, \varepsilon, t)$ is continuous on the compact set $\bar{U} \times [-\frac{\delta_1}{2}, \frac{\delta_1}{2}] \times [-T_0, T_0]$ and therefore $p(\bar{U}, [-\frac{\delta_1}{2}, \frac{\delta_1}{2}], [-T_0, T_0])$ is compact. Choose any bounded connected open set $G \subset E^N$ which contains it and again apply lemma 5.3, taking $\delta \leq \delta_1/2$.

Given a bounded connected open set $U \subset E^N$ and $T_0 > 0$, choose G and δ in accordance with lemma 5.4. Then $p(x, \varepsilon, t) \in G$ for all (x, ε, t) in $U \times (-\delta,\delta) \times [-T_0, T_0]$. For any $\tau \in E^1$, $p(x, \varepsilon, \tau)$ is analytic in (x, ε) on $G \times (-\delta,\delta)$. Therefore (see [11], pp. 33 and 35) $p(p(x, \varepsilon, t), \varepsilon,\tau)$ is analytic on $U \times (-\delta, \delta)$ for all $t \in [-T_0, T_0]$, $\tau \in E^1$, and the power series expansion of this composite function can be obtained by formal substitution. This implies the convergence of (4.30). It follows from lemma 4.3 that the relations (4.32) are satisfied, and therefore $p(x, \varepsilon, t)$ satisfies (4.29) for all $(x, \varepsilon) \in U \times (-\delta, \delta)$, $t \in [-T_0, T_0]$, and $\tau \in E^1$. Differentiation of (4.29) with respect to τ yields

$$p'(p(x, \varepsilon, t), \varepsilon, \tau) = p'(x, \varepsilon, t + \tau)$$

and setting $\tau = 0$ we obtain

$$p'(p(x, \varepsilon, t), \varepsilon, 0) = p'(x, \varepsilon, t)$$

Hence $p(x, \varepsilon, t)$ is the solution of the initial value problem

$$p' = \tilde{g}(p, \varepsilon)$$

(5.12)

$$p(x, \varepsilon, 0) = x$$

for $(x, \varepsilon) \in U \times (-\delta, \delta)$ and $t \in [-T_0, T_0]$, where

(5.13) $$\tilde{g}(x, \varepsilon) = p'(x, \varepsilon, 0)$$

According to (5.11), $p(x, \varepsilon, 1) = h(x, \varepsilon)$. So far, T_0 has been taken to be any positive number. Now require $T_0 > 1$. Then $p(x, \varepsilon, t)$ is a solution of (5.13) for $0 \leq t \leq 1$, $(x, \varepsilon) \in U \times (-\delta, \delta)$, and so $p(x, \varepsilon, t)$ satisfies (4.7) and (4.8) with $T = 1$ and $g = \tilde{g}$. However, $\tilde{g}(x, \varepsilon)$ is defined by (5.13) on all of $G \times (-\delta, \delta)$. It will now be shown that $p(x, \varepsilon, t)$ satisfies (5.12) for all $(x, \varepsilon) \in G \times (-\delta, \delta)$ if $|t|$ is sufficiently small. That is, given any $(x_0, \varepsilon) \in G \times (-\delta, \delta)$, there exists $\tau_0 > 0$ such that $p(x_0, \varepsilon, t)$ satisfies (5.12) for $|t| < \tau_0$.

Given $(x_0, \varepsilon) \in G \times (-\delta, \delta)$, there exists a connected open set G_1 such that $\bar{G}_1 \subset G$, $x_0 \in G_1$, and $G_1 \cap U \neq \emptyset$. (Since G is connected, x_0 can be connected to a point in U by a compact polygonal arc $A \subset G$. Then G_1 can be taken to be a ρ-neighborhood of A for ρ sufficiently small.) For fixed ε,

$p(x, \varepsilon, t)$ is continuous in (x, t) on $\bar{G}_1 \times E^1$ and $p(x, \varepsilon, 0) = x$. Therefore, given $x \in \bar{G}_1$, there exist $\tau_x > 0$ and a connected open set $H(x)$ containing x such that $p(H(x), \varepsilon, t) \subset G$ if $|t| < \tau_x$. But $\bar{G}_1$ is compact and so it can be covered by a finite number of these neighborhoods:

$$\bar{G}_1 \subset \bigcup_{i=1}^{k} H(x_i)$$

Let $\tau_0 = \min. (\tau_{x_1}, \ldots, \tau_{x_k})$. Then $\tau_0 > 0$ and $p(\bar{G}_1, \varepsilon, t) \subset G$ for all $\tau \in (-\tau_0, \tau_0)$. Hence, for the given ε, if $|t| < \tau_0$ then $p'(x, \varepsilon, t)$ and $\tilde{g}(p(x, \varepsilon, t), \varepsilon)$ are analytic in x on the connected open set G_1, and they are equal on $G_1 \cap U$. Therefore ([11], pp. 34 and 35) they are equal on all of G_1 and, in particular, at x_0.

The function $p(x, \varepsilon, t)$ is defined by (5.10) for all t. It follows from well known extension theorems for ordinary differential equations that $p(x, \varepsilon, t)$ satisfies (5.12) on any open interval $\alpha < t < \beta$ $(\alpha < 0 < \beta)$ on which it remains in G. Finally, in view of (5.11), we see that given $T > 0$, the function

$$v(x, \varepsilon, t) = p(x, \varepsilon, \frac{t}{T})$$

satisfies (4.7) and (4.8) with

$$g(x, \varepsilon) = \frac{1}{T} p'(x, \varepsilon, 0)$$

for any $(x, \varepsilon) \in G \times (-\delta, \delta)$ for which $v(x, \varepsilon, t) \in G$ if $0 \leq t \leq T$. The following theorem has now been established.

Theorem 5.1 If $h(x, \varepsilon)$ satisfies hypotheses (i), (ii), and (iii), then given a bounded connected open set $U \subset E^N$, $T > 0$, and $T_0 > T$, there exist a bounded connected open set $G \subset E^N$ which contains U, a positive number δ, and a function $g(x, \varepsilon)$ on $G \times (-\delta, \delta)$ into E^N which have the following properties. The function $g(x, \varepsilon)$ is analytic on $G \times (-\delta, \delta)$, $g(x, 0) = 0$, and given any $x_0 \in G$ there exists $\rho(x_0) > 0$ such that the power series expansion of $g(x, \varepsilon)$ in the $N + 1$ variables $x^{(1)} - x_0^{(1)}, \ldots, x^{(N)} - X_0^{(N)}$, ε

converges for $\|x-x_0\| < \rho(x_0)$, $|\varepsilon| < \delta$. Given any $(x, \varepsilon) \in G \times (-\delta,\delta)$, let $v(x, \varepsilon, t)$ be the solution of (4.7). If $x \in U$ then $v(x, \varepsilon, t)$ exists for $|t| \leq T_0$. Given any $(x, \varepsilon) \in G \times (-\delta,\delta)$ such that $v(x, \varepsilon, t)$ exists for $0 \leq t \leq T$, we have $v(x, \varepsilon, T) = h(x, \varepsilon)$.

We conclude this section with an examination of class of functions $h(x, \varepsilon)$ which satisfy hypotheses (i), (ii), and (iii). First it will be shown that if $h(x, \varepsilon)$ is linear and satisfies (i) and (ii), then it satisfies (iii). The linear functions which satisfy (i) and (ii) are of the form

$$h(x, \varepsilon) = A(\varepsilon)\, x \tag{5.14}$$

where $A(\varepsilon)$ is an $N \times N$ matrix, analytic in ε with an expansion of the form

$$A(\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n A_n \tag{5.15}$$

where the A_N are real constant matrices and $A_0 = I$. It is assumed that (5.15) has nonzero radius of convergence and so there exists $\delta > 0$ such that (5.15) converges for $\varepsilon \in \sigma = \{\varepsilon \in C^1 \mid |\varepsilon| < \delta\}$. Then $h(x, \varepsilon)$ satisfies (i) and (ii) with $V = C^N \times \sigma$. For any $N \times N$ matrix B let

$$\|B\| = \left\{ \sum_{i,\ j=1}^{N} |b_{ij}|^2 \right\}^{1/2}$$

Then $\|AB\| \leq \|A\|\ \|B\|$ and $\|Bx\| \leq \|B\|\ \|x\|$ for $x \in C^N$.

It follows from (4.14) and (5.14) that

$$h(x, \varepsilon, k) = A(\varepsilon)^k\, x, \quad k = 1, 2, \ldots \tag{5.16}$$

From (5.15),

$$A(\varepsilon)^k = \sum_{n_1,\ldots,n_k=0}^{\infty} \varepsilon^{n_1+\ldots+n_k} A_{n_1}\cdots A_{n_k}$$

and collecting powers of ε we obtain

$$A(\varepsilon)^k = \sum_{m=0}^{\infty} \varepsilon^m B_{mk} \tag{5.17}$$

where

$$B_{mk} = \sum_{n_1+\dots n_k=m}' A_{n_1}\cdots A_{n_k} \tag{5.18}$$

where the summation $\sum\limits_{n_1+\dots+n_k=m}$ extends over all k-vectors $(n_1,\dots,n_k)$ in which the n_j are nonnegative integers and $n_1+\dots+n_k=m$. It follows that

$$h_m(x,\ k) = B_{mk}\ x \tag{5.19}$$

Given real (or complex) numbers a_n, a necessary and sufficient condition for the series

$$\sum_{n=0}^{\infty} a_n \varepsilon^n$$

to have nonzero radius of convergence is the existence of a constant $M > 0$ such that $|a_n| \leq M^n$, $n = 1, 2, \dots$. Applying this condition to each element of the matrices A_n and noting that $\|A_0\| = \sqrt{N}$, we see that there exists $M_1 > 0$ such that

$$\|A_n\| \leq M_1^n \sqrt{N}\ ,\ n = 0,\ 1,\ \dots$$

Therefore, if $n_1 + \dots + n_k = m$, then $\|A_{n_1}\cdots A_{n_k}\| \leq N^{k/2} M_1^m$, and so each term in (5.18) is dominated by $N^{k/2} M_1^m$. There are $\binom{m+k-1}{k-1}$ terms in this sum and hence

$$\|B_{mk}\| \leq \binom{m+k-1}{k-1} N^{k/2} M_1^m$$

Let $M_2 = M_1 \sqrt{N}$; then $N^{k/2} M_1^m \leq M_2^m$, $k = 1, \ldots, m$. Also,

$$\binom{m+k-1}{k-1} < 2^{m+k-1} < 2^{2m} , \quad k = 1, \ldots, m .$$

Hence $\|B_{mk}\| < (4M_2)^m$, $k = 1, \ldots, m$, and $\|h_m(x,k)\| \leq (4M_2)^m \|x\| \leq [4M_2(\|x\| + 1)]^m$, $k = 1, \ldots, m$. This implies (iii).

A nonlinear function $h(x, \varepsilon)$ which satisfies (i), (ii) and (iii) will now be constructed. Let $N = 1$ and $P(x) = x + x^3$. Then $P'(x) = 1 + 3x^2 > 0$ for real x, and it follows that $P(x)$ has a single valued analytic inverse, call it $Q(y)$, on some complex neighborhood of the real axis ($Q(y)$ can be written explicitly in terms of radicals). Thus

$$Q(y) + Q(y)^3 = y \tag{5.20}$$

and $Q(y)$ is real if y is real. Let

$$v(x, \varepsilon, t) = P[Q(x) + \varepsilon t]$$

For real x and ε, $v(x, \varepsilon, t)$ is a dynamical system, and defining $h(x, \varepsilon) = v(x, \varepsilon, 1)$ it follows that $h(x, \varepsilon)$ satisfies (ii) and $h(x, \varepsilon, k) = v(x, \varepsilon, k)$. Expanding the above expression for $v(x, \varepsilon, t)$ with the use of (5.20) we obtain

$$v(x, \varepsilon, t) = x + [1+3Q(x)^2]\varepsilon t + 3Q(x)\varepsilon^2 t^2 + \varepsilon^3 t^3$$

Then

$$h(x, \varepsilon, k) = x + [1+3Q(x)^2]\varepsilon k + 3Q(x)\varepsilon^2 k^2 + \varepsilon^3 k^3$$

and it follows that $h(x, \varepsilon)$ can be extended to a complex neighborhood of E^2, the $h_n(x, k)$ can be extended to a complex neighborhood of E^1, and $h_n(x, k) = 0$ if $n > 3$ for all $k = 1, 2, \ldots$. Therefore $h(x, \varepsilon)$ satisfies hypotheses (i) and (iii). More generally, let $P(x)$ be any function which is analytic and has an analytic inverse, $Q(y)$, on some complex neighborhood of the real axis and which is real for real x. Let $h(x, \varepsilon) = P[Q(x) + \varepsilon]$ whereby $h(x, \varepsilon, k) = P[Q(x) + \varepsilon k]$. Then

$$h_n(x, k) = \frac{1}{n!} P^{(n)} [Q(x)]k^n$$

There exists a constant $M_1 > 0$ such that

$$\frac{n^n}{n!} \leq M_1^n , \quad n = 1, 2, \dots$$

and therefore $h(x, \varepsilon)$ satisfies (iii) provided there exists $M_2 > 0$ such that

$$|P^{(n)}(y)| \leq M_2^n , \quad n = 1, 2, \dots$$

The next, and last, example shows that the class of functions $h(x, \varepsilon)$ which satisfy (i), (ii), (iii) contains functions which grow rapidly with $\|x\|$. Let $N = 2$, $v = (v_1, v_2)$, $x = (x_1, x_2)$, and let $\ell(x)$ be any entire function on C^1. Consider the autonomous system

$$(5.21) \qquad \begin{aligned} v_1' &= -\varepsilon\, \ell(v_1^2 + v_2^2) v_2 \\ v_2' &= \varepsilon\, \ell(v_1^2 + v_2^2) v_1 \end{aligned}$$

for $(v, \varepsilon) \in C^2 \times C^1$. The solutions satisfy the condition $v_1^2 + v_2^2 = \text{constant}$, and so the solution of (5.21) which satisfies the initial condition $v(x,\varepsilon,0) = x$ is easily seen to be

$$v_1(x, \varepsilon, t) = x_1 \cos[\ell(x_1^2+x_2^2)\, \varepsilon t] - x_2 \sin[\ell(x_1^2+x_2^2)\, \varepsilon t]$$

$$v_2(x, \varepsilon, t) = x_1 \sin[\ell(x_1^2+x_2^2)\, \varepsilon t] + x_2 \cos[\ell(x_1^2+x_2^2)\, \varepsilon t]$$

Let $h(x, \varepsilon) = v(x, \varepsilon, 1)$. Then $h(x, \varepsilon, k) = v(x, \varepsilon, k)$ and it is readily seen from the power series expansions of the above functions that $h(x, \varepsilon)$ satisfies (i), (ii), (iii) with $V = C^{N+1}$. If x and ε are real, then $h(x, \varepsilon)$ is bounded for all x. But if ε is pure imaginary (and fixed) and x is real, then $\|h(x, \varepsilon)\| \to \infty$ to the order

$$\|x\| e^{\ell(\|x\|^2)|\varepsilon|}$$

as $\|x\| \to \infty$.

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APPENDIX

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Are the Perturbation Expansions for the Ground State of Helium the Same for Hartree and Hartree-Fock Model Hamiltonians?*†

R. J. Dickson and J. Sokoloff
Lockheed Palo Alto Research Laboratory, Palo Alto, California
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The Weiss-Martin variation-perturbation calculation for the first-order function in the perturbation expansion of the ground-state function of the helium atom, ostensibly using a Hartree-Fock model Hamiltonian, actually uses a Hartree model Hamiltonian. We show that the equation obtained for the first-order function using a Hartree model Hamiltonian and the corresponding equation obtained using a Hartree-Fock model Hamiltonian have no common solution. Thus, no conclusion may be drawn about the Hartree-Fock approximation on the basis of the Weiss-Martin paper.

I. INTRODUCTION

THAT the Hartree and Hartree-Fock equations are the same for the ground-state orbital of helium is a commonplace among atomic physicists. At the risk of being considered pedantic, we contend that the Hartree and Hartree-Fock equations for the ground-state orbital of helium are *not* the same; they merely have a common ground-state solution. The temptation to dispense with the exchange operator of the Hartree-Fock Hamiltonian and use the far simpler Hartree Hamiltonian in applications involving the ground state of helium has proven irresistible.

As we shall see, the form of the Hamiltonian can be all-important. In a variation-perturbation calculation of the first-order wave function for the ground state of helium, Weiss and Martin[1] (WM) ostensibly use the Hartree-Fock model Hamiltonian. In the actual calculation, WM use the Hartree rather than the Hartree-Fock operator. We shall show that the equation obtained for the first-order function in the perturbation expansion of the ground-state function of helium using a Hartree model Hamiltonian and the corresponding equation obtained using a Hartree-Fock model Hamiltonian have no common solution.

II. THE PERTURBATION EXPANSION

The nonrelativistic Hamiltonian for the helium isoelectronic sequence is given by

$$\mathcal{H}=-\tfrac{1}{2}\nabla_1^2-\frac{Z}{r_1}-\tfrac{1}{2}\nabla_2^2-\frac{Z}{r_2}+\frac{1}{r_{12}} \tag{1}$$

in atomic units.

We define Coulomb and exchange operators, respectively,

$$J_\phi(1)\theta(1)=\theta(1)\int d\tau_2\frac{|\phi(2)|^2}{r_{12}}, \tag{2a}$$

$$K_\phi(1)\theta(1)=\phi(1)\int d\tau_2\frac{\phi^*(2)\theta(2)}{r_{12}}. \tag{2b}$$

The Hartree and Hartree-Fock equations for the ground-state orbital of helium are, respectively,

$$H^{\mathrm{H}}(1)\phi(1)=\left[-\tfrac{1}{2}\nabla_1^2-\frac{Z}{r_1}+J_\phi(1)\right]\phi(1)=\eta\phi(1), \tag{3a}$$

$$H^{\mathrm{HF}}(1)\phi(1)=\left[-\tfrac{1}{2}\nabla_1^2-\frac{Z}{r_1}+2J_\phi(1)-K_\phi(1)\right]\phi(1)=\eta\phi(1), \tag{3b}$$

where η is the orbital energy.

We may rewrite the Hamiltonian of the system so as to give either a Hartree or Hartree-Fock model Hamiltonian as follows:

$$\mathcal{H}=H^{\mathrm{H}}(1)+H^{\mathrm{H}}(2)+V^{\mathrm{H}}(1,2), \tag{4a}$$

$$V^{\mathrm{H}}=(1/r_{12})-J_\phi(1)-J_\phi(2); \tag{4b}$$

or

$$\mathcal{H}=H^{\mathrm{HF}}(1)+H^{\mathrm{HF}}(2)+V^{\mathrm{HF}}(1,2), \tag{5a}$$

$$V^{\mathrm{HF}}=(1/r_{12})-2J_\phi(1)+K_\phi(1)-2J_\phi(2)+K_\phi(2). \tag{5b}$$

The ground-state wave function for the helium atom is given by the product

$$\psi_0(1,2)=\phi(1)\phi(2),$$

where we have suppressed the antisymmetric spin function and ψ_0 is a solution of

$$[H^{\mathrm{H}}(1)+H^{\mathrm{H}}(2)]\psi_0(1,2)=[H^{\mathrm{HF}}(1)+H^{\mathrm{HF}}(2)]\psi_0(1,2)=2\eta\psi_0(1,2). \tag{6}$$

We now let

$$H=H^{\mathrm{H}}(1)+H^{\mathrm{H}}(2), \tag{7a}$$

$$V=V^{\mathrm{H}}, \tag{7b}$$

or

$$H=H^{\mathrm{HF}}(1)+H^{\mathrm{HF}}(2), \tag{8a}$$

$$V=V^{\mathrm{HF}} \tag{8b}$$

and develop the standard perturbation expansions.

Let

$$\Psi=\psi_0+\psi_1+\psi_2+\cdots, \tag{9a}$$

$$E=\epsilon_0+\epsilon_1+\epsilon_2+\epsilon_3+\cdots, \tag{9b}$$

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where ψ_n's are solutions of the equations

$$(H-\epsilon_0)\psi_1=\epsilon_1\psi_0-V\psi_0, \tag{10a}$$

$$(H-\epsilon_0)\psi_2=\epsilon_2\psi_0+\epsilon_1\psi_1-V\psi_1, \tag{10b}$$

$$\vdots$$

and the energies are given by

$$\epsilon_0=2\eta, \tag{11a}$$

$$\epsilon_1=\langle\psi_0|V|\psi_0\rangle, \tag{11b}$$

$$\epsilon_2=\langle\psi_0|V|\psi_1\rangle, \tag{11c}$$

$$\vdots$$

We have assumed $\langle\psi_0|\psi_1\rangle=0$ in Eqs. (11).

III. FIRST-ORDER EQUATIONS

Let us assume the solutions to Eq. (10a) are the same for the two model Hamiltonians. We can then write the two equations in the form

$$[H^{\mathrm{H}}(1)+H^{\mathrm{H}}(2)-\epsilon_0]\psi_1=\epsilon_1{}^{\mathrm{H}}\psi_0-V^{\mathrm{H}}\psi_0, \tag{12a}$$

$$[H^{\mathrm{HF}}(1)+H^{\mathrm{HF}}(2)-\epsilon_0]\psi_1=\epsilon_1{}^{\mathrm{HF}}\psi_0-V^{\mathrm{HF}}\psi_0. \tag{12b}$$

The right-hand sides are identical, since

$$V^{\mathrm{H}}\psi_0=V^{\mathrm{HF}}\psi_0 \tag{13}$$

and

$$\epsilon_1=\langle\psi_0|V|\psi_0\rangle.$$

Taking the difference of Eqs. (12), we have

$$\psi_1(1,2)\left\{\int\frac{|\phi(3)|^2}{r_{13}}d\tau_3+\int\frac{|\phi(3)|^2}{r_{23}}d\tau_3\right\}-\phi(1)\int\frac{\phi^*(3)\psi_1(3,2)}{r_{13}}d\tau_3-\phi(2)\int\frac{\phi^*(3)\psi_1(1,3)}{r_{23}}d\tau_3=0. \tag{14}$$

We wish to show that any continuous, symmetric, square-integrable solution of this equation is of the form

$$\psi_1(1,2)=\alpha\phi(1)\phi(2), \tag{15}$$

where α is a constant. The argument is then completed by showing that a solution of this form is not an acceptable solution to one of Eqs. (12).

Equation (14) may be written in operator form as

$$\mathfrak{L}\psi_1(1,2)=0, \tag{16}$$

where $\mathfrak{L}$ is defined by

$$\mathfrak{L}\psi_1(1,2)=\int\phi^*(3)\frac{[\phi(3)\psi_1(1,2)-\phi(1)\psi_1(3,2)]}{r_{13}}d\tau_3+\int\phi^*(3)\frac{[\phi(3)\psi_1(1,2)-\phi(2)\psi_1(1,3)]}{r_{23}}d\tau_3. \tag{17}$$

Now the inner product

$$\langle\psi_1|\mathfrak{L}\psi_1\rangle=\iiint\frac{|\phi(3)|^2|\psi_1(1,2)|^2-\phi^*(3)\psi_1{}^*(1,2)\varphi(1)\psi_1(3,2)}{r_{13}}d\tau_1d\tau_2d\tau_3+\iiint\frac{|\phi(3)|^2|\psi_1(1,2)|^2-\phi^*(3)\psi_1{}^*(1,2)\phi(2)\psi_1(1,3)}{r_{23}}d\tau_1d\tau_2d\tau_3 \tag{18}$$

$$=\frac{1}{2}\iiint\frac{|\phi(3)\psi_1(1,2)-\phi(1)\psi_1(3,2)|^2}{r_{13}}d\tau_1d\tau_2d\tau_3+\frac{1}{2}\iiint\frac{|\phi(3)\psi_1(1,2)-\phi(2)\psi_1(1,3)|^2}{r_{23}}d\tau_1d\tau_2d\tau_3, \tag{19}$$

as may be verified by expanding the right-hand side and using symmetry.

Hence

$$\langle\psi_1|\mathfrak{L}\psi_1\rangle\geq 0$$

$=0$ if, and only if, both expressions in absolute signs, vanish almost everywhere. (20)

Since we assume ϕ and ψ_1 are continuous functions, the expressions in absolute signs will vanish almost everywhere only if they vanish identically, i.e., only if

$$\phi(3)\psi_1(1,2)\equiv\phi(1)\psi_1(3,2), \tag{21a}$$

and

$$\phi(3)\psi_1(1,2)\equiv\phi(2)\psi_1(1,3). \tag{21b}$$

Let β denote a point such that $\phi(\beta)\neq 0$. Then from the first of these two identities we have

$$\psi_1(1,2)=[\psi_1(\beta,2)/\phi(\beta)]\phi(1). \tag{22a}$$

From the second we have

$$\psi_1(\beta,2)=[\psi_1(\beta,\beta)/\phi(\beta)]\phi(2), \tag{22b}$$

and combining

$$\psi_1(1,2)=[\psi_1(\beta,\beta)/\phi^2(\beta)]\phi(1)\phi(2)$$
$$=\alpha\phi(1)\phi(2). \tag{23}$$

Hence, the first part of the argument is completed:

$$\mathfrak{L}\psi_1=0\Rightarrow\langle\psi_1|\mathfrak{L}\psi_1\rangle=0\Rightarrow\psi_1(1,2)=\alpha\phi(1)\phi(2).$$

Substituting this solution of Eq. (14) into Eq. (12a), we find

$$(V^{\mathrm{H}}-\epsilon_1)\psi_0=0 \tag{24}$$

and, since $\psi_0\not\equiv 0$, this implies

$$\frac{1}{r_{12}}-\int\frac{|\phi(3)|^2}{r_{13}}d\tau_3-\int\frac{|\phi(3)|^2}{r_{23}}d\tau_3=\epsilon_1, \tag{25}$$

an obvious absurdity. Thus, the argument is completed. The only solution of Eq. (14) is not a solution of Eq. (12a) and thus Eqs. (12a) and (12b) have no common solution.

IV. DISCUSSION

WM characterize their results for the energy of helium through third order as "somewhat discouraging" and a number of physicists have taken the WM calculation to indicate that the Hartree-Fock approximation is a poor zeroth-order approximation for perturbation theory. We have shown, on the contrary, that *no* conclusion may be drawn about the Hartree-Fock approximation on the basis of the WM paper. WM used the Hartree, rather than the Hartree-Fock, approximation and our proof in Sec. III indicates that the first-order functions for the two models are different.

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